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RESEARCH ARTICLE

Bound State Solution of the Klein–Gordon Equation for the Modified Screened Coulomb Plus Inversely Quadratic Yukawa Potential through Formula Method

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ABSTRACT

We present solution of the Klein–Gordon equation for the modified screened Coulomb potential (Yukawa) plus inversely quadratic Yukawa potential through formula method. The conventional formula method which constitutes a simple formula for finding bound state solution of any quantum mechanical wave equation, which is simplified to the form; $\psi''(s)+\frac{(k_1-k_2s)}{s(1-k_3s)}\psi'(s)+\frac{(As^2+Bs+c)}{s^2(1-k_3s)^2}\psi(s)=0$. The bound state energy eigenvalues and its corresponding wave function obtained with its efficiency in spectroscopy.

Key words: Bound state, inversely quadratic Yukawa, Klein–Gordon, modified screened coulomb (Yukawa), quantum wave equation

INTRODUCTION

Physicists, chemists, and other researchers in science have shown much interest in searching for the exponential-type potentials owing to reasons that most of this exponential-type potentials play an important role in physics, for example, Yukawa potential a tool used in plasma, solid-state, and atomic physics.[1] It is expressed mathematically as follows:

$$V_{yukawa}(r)=-g^2\frac{e^{-kmr}}{r}\equiv -V_0\frac{e^{-\alpha r}}{r} \quad (1)$$

$$V_{IQYP}(r)=-v_0\frac{e^{-2\alpha r}}{r^2} \quad (2)$$

Where $\alpha\to 0$ results in $V_{IQP}(r)=\frac{-V_0}{r^2}$ (3)

Respectively, where g is the magnitude sealing constant, m is the mass of the affected particle, r is the radial distance to the particles, and k is another scaling constant. The inversely quadratic potential has been used by Oyewumi *et al.*,[2] in combination with an isotropic harmonic oscillator in N-dimension spaces. Since then, several papers on the potential have appeared in the literature of Ita *et al.*,[3] first proposed by Hideki Yukawa, in 1935, on the paper titled "on the interaction of elementary particles." In his work, he explained the effect of heavy nuclei interaction on peons. According to Yukawa, he expanded that the interactions of particles are not always accompanied by the emission of light particles when heavy particles are transmitted from neutron state to proton state, but the liberated energy due to the transmission is taken up sometimes by another heavy particle which will be transformed from proton state into neutron state. Hamzavi *et al*. obtained bound state approximate analytical solutions of the Yukawa potential for any l-state through the Nikiforov-Uvarov (NU) method in which their calculations to the energy eigenvalue yielded reasonable result compared to other numerical and analytical methods. Hitler *et al.*[5] obtained the energy eigenvalue for the Yukawa potential using the generalized scaling variation method for a system with a spherically symmetric potential, Coulombic at the origin. In a nutshell, not much has been achieved on the application or use of the Yukawa potential in both relativistic and non-relativistic quantum mechanical cases. The objective of this report is to obtain band state

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solutions of the Klein–Gordon equation with the modified screened Coulomb potential plus inversely quadratic Yukawa (MSC-IQY) potential through formula method. Our work is arranged as follows; in the following section, we obtain a bound state solution, and we present discussion and conclude remarks at the end of the article.

Theoretical approach

Formula method is based on finding bound state solution of any quantum mechanical wave equation which can be simplified to the form:

$$\psi''(s)+\frac{(k_1-k_2 s)}{s(1-k_3 s)}\psi'(s)+\frac{(As^2+Bs+c)}{s^2(1-k_3 s)^2}\psi(s)=0 \quad (4)$$

The two cases where $k_3=0$ and $k\neq 3$ are studied. Furthermore, an expression for the energy spectrum and wave function in terms of generalized hypergeometric functions $2F_1(\alpha.\beta:\gamma:k_3 s)$ is derived. In a study by Falaye *et al*.,[6] the accuracy of the proposed formula has been proven in obtaining bound state solutions for some existing eigenvalue problems, which is seen to be accurate, efficient, reliable, and particularly very easy to manipulate when introduced to quite a good number of quantum mechanical potential models. In quantum mechanics, while solving relativistic and nonrelativistic quantum wave equations in the presence of central and non-central potential models, we do often come across differential equation of the form, as seen in equation (4) in which several techniques have been formulated for tackling equation (4). This includes the Feynman integral formalism, proper quantization rule, NU method, asymptotic iteration method (AIM), exact quantization rule, and generalized pseudospectral method. Falaye[7] also proposed that the energy eigenvalues and the corresponding wave function can be obtained using the following formulas with regard to this approach (formula method).

$$k_4+k_5=\frac{1-2n}{2}-\frac{1}{2k_3}\left(k_2-\sqrt{(k_3-k_2)^2-4A}\right) \quad (5)$$

Or more explicitly as,

$$\left[\frac{k_4^2-k_5^2-\left[\frac{1-2n}{2}-\frac{1}{2k_3}\left(k_2-\sqrt{(k_3-k_2)^2-4A}\right)\right]^2}{2\left[\frac{1-2n}{2}-\frac{1}{2k_3}\left(k_2-\sqrt{(k_3-k_2)^2-4A}\right)\right]}\right]$$
$$-k_5^2=0,\quad k_3\neq 0 \quad (6)$$

and

$$\psi(s)=N_n S^{k_4}(1-k_3 s)^{k_5}\ {}_2F_1(-n,n+2(k_4+k_5)+\frac{k_2}{k_3}-1;\ 2k_4+k_1>k_3 s) \quad (7),$$

respectively, such that

$$k_4=\frac{(1-k_1)+\sqrt{(1-k_1)^2-4C}}{2}, k_5=\frac{1}{2}+\frac{k_1}{2}-\frac{k_2}{2k_3}+\sqrt{\left(\frac{1}{2}+\frac{k_1}{2}-\frac{k_2}{2k_3}\right)^2-\left(\frac{A}{k_3^2}+\frac{B}{k_3}+C\right)}$$

where N_n is the normalization constant, also where $k_3\to 0$ the energy eigenvalue, and its corresponding wave function is given as,

$$\left[\frac{B-nk_2-k_2k_4}{2n+k_1+2k_4}\right]^2-k_5^2=0,\ k_3=0 \quad (8)$$

$$\psi(s)=N_n S^{k_4}e^{(-k_5 s)}\ {}_1F_1[-n;2k_4+k_1;(2k_5+k_2)s] \quad (9),$$

respectively.

Furthermore, for some other cases, where $k_3=0$, $k_3\neq 0$, and $k_5=\sqrt{-A}$. In this approach, analysis on the asymptotic behavior at the origin is first noted and at infinity for the finiteness of the solution. Considering the solution of equation (4) where $s\to 0$, it is seen that

$$\psi(s)=S^{k_4}\ \text{where}\ k_4=\frac{(1-k_1)+\sqrt{(1-k_1)^2-4C}}{2} \quad (10)$$

Furthermore, when $s\to 1/k_3$ the solution of equation 4 is $\psi(s)=(1-k_3 s)^{k_5}$,

$$k_5=\frac{1}{2}+\frac{k_1}{2}-\frac{k_2}{2k_3}+\sqrt{\left[\frac{1}{2}+\frac{k_1}{2}-\frac{k_2}{2k_2}\right]^2-\left[\frac{A}{k_3^2}+\frac{B}{k_3}+C\right]} \quad (11)$$

Hence, the wave function in the intermediary region, for this task, could be seen as follows:

$$\psi(s)=s^{k_4}(1-k_3s)^{k_5}F(s) \quad (12)$$

On substituting equation (12) into equation (10), gives a differential equation in its second-order expressed as:

$$F''(s)+F'(s)\left[\frac{(2k_4+k_1)-sk_3\left(2k_4+2k_5+\frac{k_2}{k_3}\right)}{s(1-k_3s)}\right]-\left[\frac{2k_3k_4(k_4-1)+k_5k_3(2k_4+k_1)+k_4(k_1k_3+k_2)-B}{s(1-k_3s)}\right]F(s)=0 \quad (13)$$

$$F''(s)+F'(s)\left[\frac{(2k_4+k_1)-sk_3\left(2k_4+2k_5+\frac{k_2}{k_3}\right)}{s(1-k_3s)}\right]-\left[\frac{k_3(k_4+k_5)^2+\left(k_4+k_5\right)(k_2-k_3)+\frac{A}{k_3}}{s(1-k_3s)}\right]F(s)=0 \quad (14)$$

It should be noted that equation (13) is equivalent to equation (14), whereas (13) seems to be more complex during the course of the calculation. Therefore, we employ equation (14) to obtain solutions, thereby invoking the traditional method otherwise known as the functional analysis approach and the AIM. In summary, it was noted that on applying the AIM and functional analysis method to solve equation (14) so as to obtain the formula given in equation (6) yielded the same results which imply that the result is in excellent agreement.

The Klein–Gordon equation

Given the Klein–Gordon equation as:

$$\frac{1}{h^2c^2}\left[\begin{array}{l}-\left(i\frac{\partial}{\partial t}-V(r)\right)^2-\nabla^2\text{n}^2c^2\\ +(S(r)+M)^2-\frac{l(l+1)h^2C^2}{r^2}\end{array}\right]U(r)=0 \quad (15)$$

Where M is the rest mass, $\frac{i\partial}{\partial t}$ is energy eigenvalue, and V(r) and S(r) are the vector and scalar potentials, respectively.

The radial part of the Klein–Gordon equation with vector V(r) potential = scalar S(r) potential is given as:

$$\frac{d^2U(r)}{dr^2}+\frac{1}{h^2c^2}\left[\left(E^2-M^2c^2\right)-2(E+Mc)V(r)-\frac{l(l+1)h^2c^2}{r^2}\right]U(r)=0 \quad (16)$$

In atomic units, where $h=c=1$

$$\frac{d^2U(r)}{dr^2}+\left[\begin{array}{l}\left(E^2-M^2\right)-2(E+M)V(r)\\ -\frac{l(l+1)}{r^2}\end{array}\right]U(r)=0 \quad (17)$$

The Solution to the radial part of the Klein–Gordon equation for the MSCIQY potential using formula method

The sum of the MSCIQY potential is given thus,

$$V(r)=-V_1\frac{e^{-\alpha r}}{r}-V_0\frac{e^{-2\alpha r}}{r^2} \quad (18)$$

Substituting equation (18) into equation (17), we obtain:

$$\frac{d^2U(r)}{dr^2}+\left[\begin{array}{l}\left(E^2-M^2\right)-2(E+M)\\ \left(-V_1\frac{e^{-\alpha r}}{r}-V_0\frac{e^{-2\alpha r}}{r^2}\right)-\frac{l(l+1)}{r^2}\end{array}\right]U(r)=0 \quad (19)$$

Using an appropriate approximate scheme to deal with the centrifugal/inverse square term equation (19) as proposed by Greene and Aldrich[8] wherein,

$$\frac{1}{r^2} = \frac{\alpha^2}{\left(1-e^{-\alpha r}\right)^2}; \quad \frac{1}{r} = \frac{\alpha}{\left(1-e^{-\alpha r}\right)} \tag{20}$$

Which is valid for $\alpha << 1$ for a short potential and introducing a new variable of form $s = e^{-\alpha r}$;

$$\frac{d^2U(s)}{ds^2} + \frac{1}{s}\frac{(1-s)}{(1-s)}\frac{dU(s)}{ds} + \frac{1}{\alpha^2 s^2}\left[\begin{array}{l}(E^2-M^2) - 2(E+M) \\ \left(\frac{-V_0 s^2\alpha^2}{(1-s)^2} - \frac{V_1 s\alpha}{(1-s)}\right) - \frac{\alpha^2 l(l+1)}{(1-s)^2}\end{array}\right] U(s) = 0 \tag{21}$$

$$\frac{d^2U(s)}{ds^2} + \frac{1}{s}\frac{(1-s)}{(1-s)}\frac{dU(s)}{ds} + \frac{1}{s^2}\left[\begin{array}{l}\frac{E^2-M^2}{\alpha^2} + \frac{2(E+M)V_0 s^2}{(1-s)^2} \\ + \frac{2(E+M)V_1 s}{\alpha(1-s)} - \frac{l(l+1)}{(1-s)^2}\end{array}\right] U(s) = 0 \tag{22}$$

$$\frac{d^2U(s)}{ds^2} + \frac{1}{s}\frac{(1-s)}{(1-s)}\frac{dU(s)}{ds} + \frac{1}{s^2(1-s)^2}\left[\begin{array}{l}\frac{E^2-M^2}{\alpha^2}(1-2s+s^2) + 2(E+M)V_0 s^2 + \\ \frac{2(E+M)}{\alpha}V_1 s(1-s) - l(l+1)\end{array}\right] U(s) = 0 \tag{23}$$

Where $-\beta^2 = \frac{E^2-M^2}{\alpha^2} \quad B = 2\left(\frac{E+M}{\alpha}\right)V_1$ (24)

$$\frac{d^2U(s)}{ds^2} + \frac{1}{s}\frac{(1-s)}{(1-s)}\frac{dU(s)}{ds} + \frac{1}{s^2(1-s)^2}\left[\begin{array}{l}-\left(\beta^2 + B - 2V_0(E+M)\right)s^2 + \\ \left(2\beta^2 + B\right)s - \left(\beta^2 + l(l+1)\right)\end{array}\right] U(s) = 0 \tag{25}$$

Let
$$A = -\left(\beta^2 + B - 2V_0(E+M)\right) \quad B = 2\beta^2 + B \quad C = -\left(\beta^2 + l(l+1)\right) \tag{26}$$

Now, comparing equation (25) with equation (4), A, B, and C with k_1, k_2, k_3 can easily be determined. k_4 and k_5 can be obtained as

$$k_4 = \sqrt{-C}; k_4 = \sqrt{\beta^2 + l(l+1)}; \quad k_5 = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + l(l+1) - 2V_0(E+M)} \tag{27}$$

Thus, the energy eigenvalues can easily be obtained using either equation 5 or 6; equation 5 is more preferable. Hence, on substituting k_1, k_2, k_3, k_4, k_5 into equation (5), the energy eigenvalue is obtained thus,

$$\beta^2 = \left[\frac{\begin{array}{l}2V_0(E+M) - B + l(l+1) + \\ \left(n + \tfrac{1}{2} + \sqrt{l(l+1) + \tfrac{1}{4} - 2V_0(E+M)}\right)^2\end{array}}{(2n+1) + 2\sqrt{l(l+1) + \tfrac{1}{4} - 2V_0(E+M)}}\right]^2 - l(l+1)$$

The corresponding wave functions can be obtained from equation (7) by making the needful substitutions.

$$R_{ne}(s) = N_{ne} S^{\sqrt{\beta^2 + l(l+1)}} (1-s)^{\sqrt{\frac{1}{4} + l(l+1) - 2V_0(E+M)} + \frac{1}{2}} \, {}_2F_1\left(n, n + 2\left(\sqrt{\beta^2 + l(l+1)} + \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + l(l+1) - 2V_0(E+M)}\right); 2\sqrt{\beta^2 + l(l+1)} + 1, s\right)$$

Where N_{ne} is the normalization constant.

DISCUSSION

We have obtained the energy eigenvalues and the corresponding wave function using the formula method for the MSCIQY potential. As an analogy, if we set parameters $V_0 = 0$ and $V_1 \neq 0$

$$\beta^2 = \left[\frac{-B + l(l+1) + \left(n + \frac{1}{2} + \sqrt{l(l+1) + \frac{1}{4}}\right)^2}{2n + 1 + 2\sqrt{l(l+1) + \frac{1}{4}}} \right]^2 - l(l+1)$$

$$\equiv \left[\frac{-B + 2l(l+1) + \left(l^2 + l + \frac{1}{2}\right) + (2n+1)\sqrt{l(l+1) + \frac{1}{4}}}{2n + 1 + 2\sqrt{l(l+1) + \frac{1}{4}}} \right]^2 - l(l+1)$$

The above gives the energy eigenvalue of the Klein–Gordon equation for the Yukawa potential or MSC potential.

Furthermore, if we set $V_1 = 0$ and $V_0 \neq 0$

$$\beta^2 = \left[\frac{2V_0(E+M) + l(l+1) + \left(n + \frac{1}{2} + \sqrt{l(l+1) + \frac{1}{4} - 2V_0(E+M)}\right)^2}{(2n+1) + 2\sqrt{l(l+1) + \frac{1}{4} - 2V_0(E+M)}} \right]^2; -l(l+1)$$

gives the energy eigenvalue of the K.GE for the IQY potential.

CONCLUDING REMARKS

The analytical solutions of the Klein–Gordon equation for the modified screened Coulomb plus inversely quadratic Yukawa potential have been presented through formula method in which comparison to other methods used; previously, this method has been seen to be efficient, easy to manipulate, reliable, and self-explanatory. It is energy eigenvalue, and wave function has been obtained and can be employed in the analysis of spectroscopy.

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