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REVIEW ARTICLE

On Review of the Cluster Point of a Set in a Topological Space

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ABSTRACT

If X be a topological space and A subspace of X, then a point x E X is said to be a cluster point of A if every open ball centered at x contains at least one point of A different from X. In the preliminary sections, review of the interior of the set X was discussed before the major work of section three was implemented.

Key words: Set, subset, closure and interior of set, topological space, cluster point of a set

INTRODUCTION

The word "set" in this context is used to denote the collection of well-defined object, for example, a set of books or set of student and so on. Here, let our set be denoted by the capital A and the numbers of the set by any small letters of the alphabet. Hence, the $x \epsilon A$, we say A is a subset of B if every element of $A \subset B$ while $A \subseteq B$ is used for proper subsets.

$A \cup B = [x : x \in A \text{or } x \in B\}$ while $A \cap B = \{x \in A \text{and } x \in B\}$.

Definition 1.1

A subset A of a topological space X is said to be closed if the set X-A is open, for example, the subset $[a,b]$ of R is closed because it is compliment $R-[a,b] = (-\infty, a) \cup (b+\infty)$ is open.[1]

Theorem 1.1

Let X be a topological space. Then, the following conditions hold:

1. $\emptyset$ and X are closed
2. Arbitrary intersections of closed sets are closed
3. Finite unions of closed sets are closed.[2]

Theorem 1.2

Let Y be a subspace of X. Then, a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.[3]

Theorem 1.3

Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X. Proof to above three equations is in JamesMonks p. 94-95.[4]

CLOSURE AND INTERIOR OF A SET

Given a subset A of a topological space X, the interior of A is defined as the union of all open sets contained in A and the closure of A is defined as the intersection of all closed sets containing A.

The interior of A is donated by Int A and the closure of A is denoted by CL A or by $\bar{A}$; obviously, Int A is an open set and $\bar{A}$ is a X closed set. Therefore, Int $A \subset A \subset \bar{A}$

If A is open, $A =$ Int A while if A is closed $A = \bar{A}$

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Theorem 2.1

Let Y be a subspace of X and let A be a subset of Y. The closure of A in Y equals $\bar{A} \cap Y$.[5]

Theorem 2.2

Let A be a subset of the topological space X.

a) Then, $X \in \bar{A}$ if and only if every open set U is containing x interests A.
b) Supposing the topology of X is given by a basis theory $x \in \bar{A}$ if and only if every basis element B containing x interests A.[6,7]

Proof

Consider the statement (a), it is a statement of the form $P \Leftrightarrow Q$. Let us transform each implication to its contrapositive, thereby obtaining the logically equivalent statement (not P) $\Leftrightarrow$ (not Q). Written out it is the following:
$X \notin \bar{A}$ Δ there exists an open set U containing x that does not interest A.
In this form, our theory is easy to prove. If x is not in $\bar{A}$, the set $U = x - \bar{A}$ is an open set containing x that does not interest A as desired. Conversely, if there exist an open set U containing x that does not intersect A, then $X-U$ is a closed set. By definition of the must contain $\bar{A}$, the set $X-U$ must contain A; therefore, X cannot be in $\bar{A}$. Statement (b) follows readily. If every open set containing x interests A so does every basis element B containing x, because B is an open set.
Conversely, if every basis element containing x interest A so does every open set U containing x because U contains a basis element that contains x. Hence, U is an open set containing x, i.e. U is a neighborhood of x.
Therefore, if A is a subset of the topological space X, then $x \in \bar{A}$ if and only if every neighborhood of x intersects A.

CLUSTER POINT (LIMITED POINT OR ACCUMULATION POINT) OF THE SET A

This is another way of defining the closure of a set.
Definition 3.1
Let X be a topological space and A be a subspace of X. A point $x \in X$ (not necessary in A) is said to be a cluster point or a limit point or an a accumulation point of A if every open ball centered at x contains at least one point of A different from x (i.e., x is a limit point of A if $B_r(x) - \{x\} \cap A \neq 0$ for real number ($r>0$).[3]

Equivalent definition

If A is a subset of a topological space X and if x is a point of X, we say that x is a limit point (or cluster point or accumulation point of A if every neighborhood of x intersects A in some point other than x itself. Alternatively, x is a limit point of A if it belongs to the closure of $A - \{x\}$.

Another alternative definition

When $X = R$ and $A \subseteq X = R$, we say that x is a limit point of R if $\forall \delta > 0, (x-\delta, x+\delta) \Delta A\{x\} \neq 0$

Definition 3.2

The set of all limit point of A denoted by A' is called the derived set of A.[7]

Definition 3.3

A subset of A of a metric space X is said to be closed if it contains all its limit points.[4]

Theorem 3.1

Let A be a subset of the topological space X and let A' be the point of A, the $\bar{A} = A \cup A'$

Proof

If x is in A', every neighborhood of x intersects A (in a point different from X). Therefore, $x \in \bar{A}$. Hence, $A' \subset \bar{A}$. Since by definition $A \subset \bar{A}$, it follows that $A \cup A' \subset \bar{A}$ to demonstrate the reverse inclusion we let x be a point of $\bar{A}$ and show that $x \in A \cup A'$ supposed that x does not lie in A.
Since, $x \in A$, we know that every neighborhood U of x intersects A in a point different from x. Then, $x \in \bar{A}$ so that $x \in A \cup A'$, as desired.

Theorem 3.2

Let $x \in R$ and $A \subseteq R$.

I. If x has a neighborhood which only contains finitely many members of A, then x cannot be a limit point of A.

II. If x is a limit point of A, then any neighborhood of x contains infinitely many members of A.[2]

Proof I

Let U be a neighborhood of x which contains only a finite number of point A that is $U \delta A$ is finite. Suppose $U \delta A \{x\}=[y_1, y_2,..., y_n]$. We show that there exists $\delta > 0$ such that $(x-\delta, x+\delta) \cap U$ does not contain any member of $A\setminus\{x\}$. Since both $x-\delta, x+\delta$ and U neighborhood of x so is their intersection.
This will prove that there is a neighborhood of x containing no element of $A\setminus\{x\}$ hence proving x is not a limit point of A. Let $\delta = \min\{|x-y_1|,|x-y_2|,...,|x-y_n|\}$ since x is not equal to y_i, $\delta > 0$. Then, $(x-\delta, x+\delta)$ contains no point of A other than x, thus proving our claim. Proof of II follows immediately from I.

Corollary 3.3

No finite set can have a limit point.

Proof

Follows immediately from Theorem 2.2.

Theorem 3.4

Let $A \subseteq R$, then

i. $A \cup L(A) = \bar{A} = A \cup A$

ii. $L(A) \cup L(B) = L(A \cup B)$, i.e., $A' \cup B' = (A \cup B)'$

iii. $L(A)$ is closed, i.e. A' is closed.

iv. If U is open, then $L(U) = U$, i.e., $U' = U$

(i) above was clearly captured in the proof of Theorem 2.1 while (ii) and (iv) are follow ups of already proved theorems above.

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