

REVIEW ARTICLE

On Frechet Derivatives with Application to the Inverse Function Theorem of Ordinary Differential Equations

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ABSTRACT

In this paper, the Frechet differentiation of functions on Banach space was reviewed. We also investigated that it is algebraic properties and its relation by applying the concept to the inverse function theorem of the ordinary differential equations. To achieve the feat, some important results were considered which finally concluded that the Frechet derivative can extensively be useful in the study of ordinary differential equations.

Key words: Banach space, Frechet differentiability, Lipschitz function, inverse function theorem

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INTRODUCTION

The Frechet derivative, a result in mathematical analysis, is derivative usually defined on Banach spaces. It is often used to formalize the functional derivatives commonly used in physics, particularly quantum field theory. The purpose of this work is to review some results obtained on the theory of the derivatives and apply it to the inverse function theorem in ordinary differential equations.^[1-5]

DERIVATIVES**Definition 1.1 (Kaplun [1958])**

Let f be an operator mapping a Banach space X into a Banach space Y . If there exists a bounded linear operator T from X into Y such that

$$\lim_{\|\Delta x\| \rightarrow 0} \frac{\|f(x_0 + \Delta x) - f(x_0) - T(\Delta x)\|}{\|\Delta x\|} = 0 \quad (1.1)$$

or,

$$\lim_{t \rightarrow 0} \frac{\|f(x + th) - f(x)\|}{\|t\|} = T_x(h)$$

or

$$\lim_{y \rightarrow 0} \frac{\|f(x + y) - f(x) - T(y)\|}{\|y\|} = 0 \quad (1.1)$$

then P is said to be Frechet differentiable at x_0 , and the bounded linear operator

$$P(x_0) = T \quad (1.2)$$

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is called the first Frechet – derivative of f at x_0 . The limit in (1.1) is supposed to hold independent of the way that Δx approaches 0. Moreover, the Frechet differential

$$\delta f(x_0, \Delta x) = f'(x_0) \Delta x$$

is an arbitrary close approximation to the difference $f(x_0 + \Delta x) - f(x_0)$ relative to $\|\Delta x\|$, for $\|\Delta x\|$ small.

If f_1 and f_2 are differentiable at x_0 , then

$$(f_1 + f_2)'(x_0) = f_1'(x_0) + f_2'(x_0)$$

Moreover, if f_2 is an operator from a Banach space X into a Banach space Z , and f_1 is an operator from Z into a Banach space Y , their composition $f_1 \circ f_2$ is defined by

$$(f_1 \circ f_2)(x) = f_1(f_2(x)), \text{ for all } x \in X$$

We know that $f_1 \circ f_2$ is differentiable at x_0 if f_2 is differentiable at x_0 and f_1 is differentiable at $f_2(x_0)$ of Z , with (chain rule):

$$(f_1 \circ f_2)'(x) = f_1'(f_2(x_0)) f_2'(x_0)$$

To differentiate an operator f we write:

$$f(x_0 + \Delta x) - f(x_0) = T(x_0, \Delta x) \Delta x + \eta(x_0, \Delta x),$$

where $T(x_0, \Delta x)$ is a bounded linear operator for given $x_0, \Delta x$ with

$$\lim_{\|\Delta x\| \rightarrow 0} T(x_0, \Delta x) = T \tag{1.3}$$

and

$$\lim_{\|\Delta x\| \rightarrow 0} \frac{\|\eta(x_0, \Delta x)\|}{\|\Delta x\|} = 0 \tag{1.4}$$

Estimate (1.3) and (1.4) give

$$\lim_{\|\Delta x\| \rightarrow 0} T(x_0, \Delta x) = f'$$

If $T(x_0, \Delta x)$ is a continuous function of Δx in some ball $\cup(0, R), (R > 0)$, then

$$T(x_0, 0) = f'(x_0)$$

We now present the definition of a mosaic:

Higher – orderer derivatives can be defined by induction:

Definition 1.2 (Argyros [2005])

If f is $(m - 1)$ – times Frechet – differentiable ($m \geq 2$ an integer) and an m – linear operator A from X into Y exists such that

$$\lim_{\|\Delta x\| \rightarrow 0} \frac{\|f^{(m-1)}(x_0 + \Delta x) - f^{(m-1)}(x_0) - A(\Delta x)\|}{\|\Delta x\|} = 0 \tag{1.5}$$

then A is called the m – Frechet – derivative of f at x_0 , and

$$A = f^{(m)}(x_0) \tag{1.6}$$

Definition 1.3 (Koplan [1958])

Suppose $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ where U is an open set, the function f is classically differentiable at $x_0 \in U$ if

The partial derivatives of f , $\frac{\partial f_i}{\partial x_j}$ for $i = 1 \dots m$ and $j = 1 \dots n$ exist at x_0 ,

The Jacobean matrix $J(x_0) = \left[\frac{\partial f_i}{\partial x_j}(x_0) \in \mathbb{R}^{m \times n} \right]$ satisfies

$$\lim_{x \rightarrow x_0} \frac{\|f(x) - f(x_0) - J(x_0)(x - x_0)\|}{\|x - x_0\|} = 0.$$

We say that the Jacobean matrix $J(x_0)$ is the derivative of f at x_0 that is called total derivative

Higher partial derivatives in product spaces can be defined as follows:

Define

$$X_{ij} = T(X_j, X_i) \tag{1.7}$$

where X_1, X_2, \dots are Banach spaces and $T(X_j, X_i)$ is the space of bounded linear operators from X_j into X_i . The elements of X_{ij} are denoted by T_{ij} , etc. Similarly,

$$X_{ijm} = T(X_m, X_{ij}) = T(X_{jk}, X_{ij_1 j_2 \dots j_{m-1}}), \tag{1.8}$$

which denotes the space of bonded linear operators from X_{jm} into $X_{ij_1 j_2 \dots j_{m-1}}$. The elements $A = A_{ij_1 j_2 \dots j_m}$ are a generalization of m -linear operators.

Consider an operator f_1 from space

$$X = \prod_{p=1}^n X_{jp} \tag{1.9}$$

into X_i , and that f_i has a partial derivative of orders $1, 2, \dots, m-1$ in some ball $\cup(x_0, R)$, where $R > 0$ and

$$x_0 = (x_{j_1}^{(0)}, x_{j_2}^{(0)}, \dots, x_{j_n}^{(0)}) \in X$$

For simplicity and without loss of generality, we (Frechet [1906]) remember the original spaces so that

$$j_1 = 1, j_2 = 2, \dots, j_n = n$$

hence, we write

$$x_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$$

A partial derivative of order $(m-1)$ of f_i at x_0 is an operator

$$A_{iq_1 q_2 \dots q_{m-1}} = \frac{\partial^{(m-1)} f_i(x_0)}{\partial x_{q_1} \partial x_{q_2} \dots \partial x_{q_{m-1}}} \tag{1.10}$$

(in $X_{iq_1 q_2 \dots q_{m-1}}$) where

$$1 \leq q_1, q_2, \dots, q_{m-1} \leq n$$

Let $P(X_{q_m})$ denote the operator from X_{q_m} into $X_{i_1 q_2 \dots q_{m-1}}$ obtained by letting

$$x_j = x_j^{(0)}, \quad j \neq q_m$$

for some $q_m, 1 \leq q_m \leq n$. Moreover, if

$$P'(x_{q_m}^{(0)}) = \frac{\partial}{\partial x_{q_m}} \cdot \frac{\partial^{m-1} f_i(x_0)}{\partial x_{q_1} \partial x_{q_2} \dots \partial x_{q_{m-1}}} = \frac{\partial^m f_i(x_0)}{\partial x_{q_1} \dots \partial x_{q_m}}, \quad (1.11)$$

exists, it will be called the partial Frechet – derivative of order m of f_i with respect to x_{q_1}, \dots, x_{q_m} at x_0

Furthermore, if f_i is Frechet – differentiable m times at x_0 , then

$$\frac{\partial^m f_i(x_0)}{\partial x_{q_1} \dots \partial x_{q_m}} x_{q_1} \dots x_{q_m} = \frac{\partial^m f_i(x_0)}{\partial x_{s_1} \partial x_{q_2} \dots \partial x_{s_m}} x_{s_1} \dots x_{s_m} \quad (1.12)$$

For any permutation s_1, s_2, \dots, s_m of integers q_1, q_2, \dots, q_m and any choice of point x_{q_1}, \dots, x_{q_m} , from X_{q_1}, \dots, X_{q_m} , respectively. Hence, if $f = (f_1, \dots, f_t)$ is an operator from $X = X_1 + X_2 + \dots + X_n$ into $Y = Y_1 + Y_2 + \dots + Y_t$ then

$$f^{(m)}(x_0) = \left(\frac{\partial^m f_i}{\partial x_{j_1} \dots \partial x_{j_m}} \right)_{x=x_0} \quad (1.13)$$

$i=1, 2, \dots, t, j_1, j_2, \dots, j_m = 1, 2, \dots, n$ is called the m – Frechet derivative of F at $x_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$.

Intention^[6]

In this subsection, we due to (Pantryagin [1962]) state results concerning the mean value theorem, Taylor's theorem and Riemannian integration without the proofs. Hence the mean value theorem for differentiable real functions f is stated as follows:

$$f(b) - f(a) = f'(c)(b - a),$$

Where $c \in (a, b)$, does not hold in a Banach space setting. However, if F is a differentiable operator between two Banach spaces X and Y , then

$$\|f(x) - f(y)\| \leq \sup_{x \in L(x, y)} \|f'(\bar{x})\| \cdot \|x - y\|$$

where

$$T(x, y) = \{z : z = \lambda y + (1 - \lambda)x, 0 \leq \lambda \leq 1\}$$

Set

$$z(\lambda) = \gamma_y + (1 - \lambda)x, 0 \leq \lambda \leq 1,$$

Divide the interval $0 \leq \lambda \leq 1$ into n subintervals of lengths $\Delta \lambda_i, i = 1, 2, \dots, n$, choose points λ_i inside corresponding subintervals and as in the real Riemann integral consider sums

$$\sum_{\sigma} f(\lambda_i) \Delta \lambda_i = \sum_{i=1}^n f(\lambda_i) \Delta \lambda_i$$

where σ is the partition of the interval, and set

$$|\sigma| = \max_{(i)} \Delta \lambda_i$$

Definition 1.4 (Ince [1956])

If

$$S = \lim_{|\sigma| \rightarrow 0} \sum_{\sigma} f(\lambda_i) \Delta \lambda_i$$

exists, then it is called the Riemann integral from $f(\lambda)$ on $[0,1]$, denoted by

$$S = \int_0^1 f(\lambda) d\lambda = \int_x^y f(\lambda) d\lambda$$

Definition 1.5^[7]

A bounded operator $P(\lambda)$ on $[0,1]$ such that the set of points of discontinuity is of measure zero is said to be integrable on $[0,1]$.

Theorem 1.1 (Day [1973])

If F is m – times Frechet – differentiable in $U(x_0, R), R > 0$, and $f^{(m)}(x)$ is integrable from x to any $\in U(x_0, R)$ then

$$f(y) = f(x) + \sum_{n=1}^{m-1} \frac{1}{n!} f^{(n)}(x)(y-x)^n + R_m(x, y), \quad (1.14)$$

$$\left\| f(y) - \sum_{n=0}^{m-1} \frac{1}{n!} f^{(n)}(x)(y-x)^n \right\| \leq \sup_{\bar{x} \in L(x, y)} \|f^{(m)}(\bar{x})\| \frac{\|y-x\|^m}{m!}, \quad (1.15)$$

where

$$R_m(x, y) = \int_0^1 f^{(m)}(\lambda y + (1-\lambda)x) (y-x)^m \frac{(1-\lambda)^{m-1}}{(m-1)!} d\lambda \quad (1.16)$$

THEOREMS AND PROPERTIES^[8,9]

Definition 2.1 (Arthanasius [1973])

Assume that X, Y are Banach spaces, S is an open subset of X and $f : S \rightarrow Y$, is continuous and Frechet differentiable at every point of S , Moreover, assume that for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

- (i) If $x_1, x_2, \in S$ with $\|x_1 - x_2\| \leq (C \epsilon)$ then $\| [f'(x_1) - f'(x_2)]h \| \leq \epsilon \forall \|h\| \in X$
- (ii) $\| \omega(x, L) \| \leq \omega[h]$ for every $x \in S, h \in X$ with $h \leq b(\epsilon)$ then f is called C – differentiable on S

Theorem 2.1 (Arthanasius [1973])

Let f be a convex function defined on an open convex subset X of a Banach space X that is continuous at $x \in X$. Then, f is Frechet differentiable at x iff $\lim_{t \rightarrow 0} \frac{\|f(x+th) + f(x-th) - 2f(x)\|}{\|t\|} = D$ uniformly for $h \in S_x$

Remark 2.1

Obviously, we have earlier seen that Frechet differentiability has additive property and the product of two Frechet differentiable functions is Frechet differentiable function (the function that is Frechet differentiable is continuous and therefore locally bounded). Now, we use boundedness of Frechet derivative and triangle inequality. A function which is Frechet differentiable at a point is continuous there.

THE INVERSE FUNCTION THEOREM

An inverse function theorem is an important tool in the theory of differential equations. It ensures the existence of the solution of the equation $Tx = y$. Although T is not assumed to be compact and the contraction principle might not be directly applicable, it is shown, in the proof of the inverse function theorem, that the contraction principle can be used indirectly if T has some appropriate differentiability properties.

Definition 3.1 (Arthanasius [1973])

Let S be an open subset of the Banach space X and let f map S into the Banach space Y . Fix a point $x_0 \in S$ and let $f(x_0) = y_0$, then, f is said to be "locally invertible at (x_0, y_0) " if there exist two numbers $\alpha > 0$, $\beta > 0$, with the following property: For every $y \in S_0(y_0)$, there exists a unique $x \in S_0(x_0)$ such that $f(x) = y$.

Lemma 3.1 (uniqueness property of Frechet derivative) (Kaplan [1958])

Let $f: S \rightarrow Y$ be given with S an open subset of the Banach space X and Y another Banach space. Suppose further that f is Frechet differentiable at $x \in S$. Then, the Frechet derivative of f at x is unique.

Proof

Suppose that $D_1(x), D_2(x)$ are Frechet derivatives of f at x with remainders $\omega_1(x, h), \omega_2(x, h)$, respectively. Then, we have

$$D_1(x)h + \omega_1(x, h) = D_2(x)h + \omega_2(x, h)$$

for every $h \in X$ with $x + h \in S_1$. Here S_1 is some open subsets of S containing x . It follows that

$$\|D_1(x)h - D_2(x)h\| / \|h\| = \|\omega_1(x, h) - \omega_2(x, h)\| \quad (3.1)$$

$$1 \leq \|\omega_1(x, h)\| / \|h\| + \|\omega_2(x, h)\| / \|h\|$$

The last member of (3.1) tends to zero as $\|h\| \rightarrow 0$.

Let

$T_x = [D_1(x) - D_2(x)]x$, $x \in X$ then T is a linear operator on X such that

$$\lim_{x \rightarrow 0} Tx / x = 0$$

Consequently, given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $\|Tx\| / \|x\| < \epsilon$ for every $x \in X$ with $\|x\| < \delta(\epsilon)$. Given $y \in X$ with $y \neq 0$, let $x = \delta(\epsilon)y / 2\|y\|$.

Then $\|x\| < \delta(\epsilon)$ and $\frac{\|Tx\|}{\|x\|} < \epsilon$ or $\|Ty\| < \epsilon\|y\|$. Since ϵ is arbitrary, we obtain $Ty = 0$ for every $y \in X$. We

conclude that $D_1(x) = D_2(x)$.

The existence of a bounded Frechet derivative f' is equivalent to the continuity of f at x . This is the content of the next lemma.

Lemma 3.2 (Argyros [2005])

Let $f : S \rightarrow Y$ be given where S is an open subset of a Banach space X and Y Another Banach space, Let f be Frechet differentiable at $x \in S$. Then, f is continuous at x if $f'(x)$ is a bounded linear operator.

Proof

Let f be continuous at $x \in S$. Then, for each $\epsilon > 0$ there exists $\delta(\epsilon) \in (0,1)$ such that

$$\|f(x+h) - f(x)\| < \epsilon/2, \|f'(x)h\| < (\epsilon/2)\|h\| < \epsilon/2$$

for all $h \in B$ with $x+h \in S$ and $\|h\| < \delta(\epsilon)$.

Therefore.

$$\|f'(x)h\| < (\epsilon/2)\|h\| + \|f(x+h) - f(x)\| < \epsilon$$

For $\|h\| < \delta(\epsilon)$, $x+h \in S$, it follows that the linear operator $f'(x)$ is continuous at the point 0.

We should note here that the magnitude of the ball S plays no role in the Frechet differentiability of f . This means that, to define the Frechet derivative, $f'(x)$ of f at x , we only need to have that certain differentiability conditions hold for all $x+h$ in a sufficiently small open neighborhood S of the point x .

We now quote a well-known theorem of functional analysis the “bounded inverse theorem.”

Theorem 3.2 (Day [1973])

Let X, Y be Banach spaces and let $T : X \rightarrow Y$ be bounded linear, one-to-one and onto. Then, the inverse T^{-1} of T is a bounded linear operator on Y .

We are ready for the inverse function theorem.

Theorem 3.3 (inverse function theorem) (Arthanasius [1973])

Let X, Y be Banach spaces and S an open subset of X Let $f : S \rightarrow Y$ be C-differentiable on S . Moreover, assume that the Frechet derivative of the function is one-to-one and onto at some point $x_0 \in S$. Then, the function f is locally invertible at the point $(x_0, f(x_0))$.

Proof

Let $D = f'(x_0)$. Then, the operator D^{-1} exists and is defined on Y . Moreover D^{-1} is bounded. Thus, the equation $f(x) = y$ is equivalent to the equation $D^{-1}f(x) = D^{-1}y$. fix $y \in Y$ and define the operator U on S as follows.

$$U_x = x + D^{-1} [y - f(x)], x \in S \tag{3.2}$$

obviously, the fixed points of the operator U are solutions to the equation $f(x) = y$. We first determine a closed ball inside S with center at x_0 , on which U is a contraction operator. To this end fix $\epsilon \in (0, 1/4\|D^{-1}\|)$ and let $\delta(\epsilon) < 0$ be such that

$$\| [f'(x_0) - f'(x_2)](x_1 - x_2) \| \leq \epsilon \|x_1 - x_2\| \quad \dots \quad (3.3)$$

$$\| \omega(x_2, x_1 - x_2) \| \leq \epsilon \|x_1 - x_2\| \quad \dots \quad (3.4)$$

For every $x_1, x_2 \in S$ with $\|x_1 - x_0\| \leq \frac{\delta(\epsilon)}{2} \|x_2 - x_0\| \leq \delta(\epsilon)/2$

This is possible by virtue of the C-differentiability of the function f . Thus, we have

$$\begin{aligned} \|Xx_1 - Xx_2\| &= \|x_1 - x_2 - D^{-1}[f(x_1) - f(x_2)]\| \\ &= \|D^{-1}f'(x_0)(x_1 - x_2) - D^{-1}f'(x_2)(x_1 - x_2) - D^{-1}\omega(x_2, x_1 - x_2)\| \\ &\leq \|D^{-1}\| \| [f'(x_0) - f'(x_2)](x_1 - x_2) \| + \|D^{-1}\| \|\omega(x_2, x_1 - x_2)\| \\ &\leq \epsilon \|D^{-1}\| \|x_1 - x_2\| + \epsilon \|D^{-1}\| \|x_1 - x_2\| \\ &\leq (1/2) \|x_1 - x_2\| \end{aligned}$$

For every $x_1, x_2 \in S$ as above. It follows that X is a contraction operator on the ball $S_0(x_0)$, where $\alpha = \delta(\epsilon)/2 < 1/(4\|D^{-1}\|)$.

Now, we determine a constant $\delta > 0$ such that $XS_0(x_0) \subset S_0(x_0)$ whenever $v \in S_0(Y_0)$ Here $v_0 = f(x_0)$. in fact we have

$$\|Xx_0 - Xx_0\| \leq \|D^{-1}\| \|y - f(x_0)\| = \|D^{-1}\| \|y - y_0\| \leq \delta(\epsilon)/4$$

whenever

$$\|y - y_0\| \leq \delta(\epsilon)/(4\|D^{-1}\|) = \beta$$

Furthermore

$$\begin{aligned} \|X - x_0\| &= \|Xx - Xx_0\| + \|Xx_0 - x_0\| \\ &\leq (1/2) \|x - x_0\| + \delta(\epsilon)/4 \leq \delta(\epsilon)/4 + \delta(\epsilon)/4 = \delta(\epsilon)/2 \end{aligned}$$

For any $x \in S_0(x_0)$. We have shown that f is locally invertible at $(x_0, f(x_0))$. and that for any $y \in S_0(y_0)$, there exists a unique $x \in S_0(x_0)$ With $f(x) = y$.

Theorem 3.4 (Ince [1956])

Let the assumption of theorem 3.3 be satisfied with the C-differentiability of the function f replaced by condition (i) of definition 3.1, then the conclusion of theorem 3.3 remains valid.

Example 3.1 (Leighton [1970]) let $J = [a, b]$ and let

$$S_r = \{x \in \mathbb{R}^n : \|x\| < r\}$$

$$S^r = [x \in C_n(J); \|\cdot\|_\infty < r]$$

Where r is a positive number.

We consider a continuous function $F : J \times \bar{S}_0 \rightarrow R^n$ and the operator $X : S^r \rightarrow C_n(J)$ defined as follows:

$$(Xx)(t) = F(t, x(t)), t \in J, \infty \in S^r$$

We first note that W is continuous on S . In fact, since F is uniformly continuous on the compact set $J \times \bar{S}_0$ for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\|F(t, x) - F(t, y)\| < \epsilon$$

For every $x, y \in \bar{S}_0$ with $\|x - y\| < \delta(\epsilon)$ and every $t \in J$. This implies that

$$\|Ux - Uy\|_\infty < \epsilon$$

Whenever, $x, y \in S^r$ with $\|x, y\|_\infty < \delta(\epsilon)$. To compute the Frechet derivative of X , we assume that the Jacobean matrix

$$F_1(t, x) = [(\partial F / \partial x)(t, x)], i = 1, 2, \dots, n$$

exists and is continuous on $J \times S_0$, then given two function

$x_0 \in S^r$ ($0 < r_1 < r$), $h \in J$ such that $x_0 + h \in C_n(J)$ $x_0 + h \in S^r$ we have

$$\begin{aligned} & \sup_{s \in I} \|F(t + h(t)) - F(t, x_0(t)) - F_1(t, x_0(t))h(t)\| \\ & \leq \sup_{s \in I} \left\{ \left\| \left[(\partial F / \partial x F(t, x_0(t))\theta, h(t)) \right] - F_1(t, x_0(t)) \right\| \|h(t)\| \right\} \end{aligned} \quad (3.5)$$

Where $\theta_i, i = 1, 2, \dots, n$ are function of t lying in the interval $(0, 1)$. In (3.5), we have used the mean value theorem for real-valued functions on S_0 as follows:

$$F_1(t, x_0(t) + h(t)) - F_1(t, x_0(t)) = \langle \nabla F_1(t_1, z_1(t))h(t) \rangle, i = 2, \dots, n$$

Where $z_1(t) = x_0(t) + \theta_1 h(t)$ form the uniform continuity of $\nabla F_1(t_1, z_1(t))h(t) > i = 1, \dots, n$ on $J \times S_1$, it follows that the Frechet derivative $X(x_0)$ exists and is a bounded linear operator given by the formula

$$[X'(x_0)h](t) = F_1(t, x_0(t))h(t) \quad (3.6)$$

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