ISSN 2581-3463

AJMS

Asian Journal of Mathematical Sciences

RESEARCH ARTICLE

# Solving High-order Non-linear Partial Differential Equations by Modified q-Homotopy Analysis Method

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**Received: 25-04-2020; Revised: 25-05-2020; Accepted: 10-06-2020**

## ABSTRACT

In this paper, modified q-homotopy analysis method (mq-HAM) is proposed for solving high-order non-linear partial differential equations. This method improves the convergence of the series solution and overcomes the computing difficulty encountered in the q-HAM, so it is more accurate than nHAM which proposed in Hassan and El-Tawil, Saberi-Nik and Golchaman. The second- and third-order cases are solved as illustrative examples of the proposed method.

**Key words:** Non-linear partial differential equations, q-homotopy analysis method, modified q-homotopy analysis method

## INTRODUCTION

Most phenomena in our world are essentially non-linear and are described by non-linear equations. It is still difficult to obtain accurate solutions of non-linear problems and often more difficult to get an analytic approximation than a numerical one of a given non-linear problem. In 1992, Liao[1] employed the basic ideas of the homotopy in topology to propose a general analytic method for non-linear problems, namely, homotopy analysis method (HAM). In recent years, this method has been successfully employed to solve many types of non-linear problems in science and engineering.[2-11] All of these successful applications verified the validity, effectiveness, and flexibility of the HAM. The HAM contains a certain auxiliary parameter $h$ which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called $h$-curve, it is easy to determine the valid regions of $h$ to gain a convergent series solution. Hassan and El-Tawil[7] presented a new technique of using HAM for solving high-order non-linear initial value problems (nHAM) by transform the nth-order non-linear differential equation to a system of $n$ first-order equations. El-Tawil and Huseen[12] established a method, namely, q-HAM which is a more general method of HAM. The q-HAM contains an auxiliary parameter $n$ as well as $h$ such that the case of $n$=1 (q-HAM; $n$=1) the standard HAM can be reached. The q-HAM has been successfully applied to numerous problems in science and engineering.[12-22] Huseen and Grace[23] presented modifications of q-HAM (mq-HAM). They tested the scheme on two second-order non-linear exactly solvable differential equations. The aim of this paper is to apply the mq-HAM to obtain the approximate solutions of high-order non-linear problems by transform the nth-order non-linear differential equation to a system of $n$ first-order equations. We note that the case of $n$=1 in mq-HAM (mq-HAM; $n$=1), the nHAM[7] can be reached.

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## ANALYSIS OF THE Q-HAM

Consider the following non-linear partial differential equation:

$$N[u(x, t)]=0 \tag{1}$$

Where, $N$ is a non-linear operator, $(x, t)$ denotes independent variables, and $u(x, t)$ is an unknown function. Let us construct the so-called zero-order deformation equation:

$$(1-nq)L[\varnothing(x, t; q)-u_0 (x, t)]=qhH(x, t)N[\varnothing(x, t; q)] \tag{2}$$

where $n\geq1$, $q\in[0, \frac{1}{n}]$ denotes the so-called embedded parameter, $L$ is an auxiliary linear operator with the property $L[f]=0$ when $f=0$, $h\neq0$ is an auxiliary parameter, $H(x, t)$ denotes a non-zero auxiliary function. It is obvious that when $q=0$ and $q=\frac{1}{n}$ Equation (2) becomes

$$\varnothing(x,t;0)=u_0(x,t) \qquad and \qquad \varnothing\left(x,t;\frac{1}{n}\right)=u(x,t) \tag{3}$$

respectively. Thus, as $q$ increases from 0 to $\frac{1}{n}$, the solution $\varnothing(x, t; q)$ varies from the initial guess $u_0 (x, t)$ to the solution $(x, t)$. We may choose $u_0 (x, t)$, $L$, $h$, $H (x, t)$ and assume that all of them can be properly chosen so that the solution $\varnothing(x, t; q)$ of Equation (2) exists for $q\in[0, \frac{1}{n}]$.

Now, by expanding $\varnothing(x, t; q)$ in Taylor series, we have

$$\varnothing(x,t;q)=u_0(x,t)+\sum_{m=1}^{+\infty}u_m(x,t)q^m \tag{4}$$

where

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m\varnothing(x,t;q)}{\partial q^m}|_{q=0} \tag{5}$$

Next, we assume that $h$, $H (x, t)$, $u_0 (x, t)$, $L$ are properly chosen such that the series (4) converges at $q=\frac{1}{n}$ and:

$$u(x,t)=\varnothing\left(x,t;\frac{1}{n}\right)=u_0(x,t)+\sum_{m=1}^{+\infty} u_m(x,t)\left(\frac{1}{n}\right)^m \tag{6}$$

We let

$$\vec{u}_r(x,t)=\{u_0(x,t),u_1(x,t),u_2(x,t),\ldots,u_r(x,t)\}$$

Differentiating equation (2) $m$ times with respect to $q$ and then setting $q=0$ and dividing the resulting equation by $m!$ we have the so-called $m^{th}$ order deformation equation

$$L\left[u_m(x,t)-k_m u_{m-1}(x,t)\right]=hH(x,t)R_m(\vec{u}_{m-1}(x,t)) \tag{7}$$

where,

$$R_m(\vec{u}_{m-1}(x,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}(N[\varnothing(x,t;q)]-f(x,t))}{\partial q^{m-1}}|_{q=0} \tag{8}$$

and

$$k_m = \begin{cases} 0 & m \le 1 \\ n & otherwise \end{cases} \tag{9}$$

It should be emphasized that $u_m(x, t)$ for $m \geq 1$ is governed by the linear Equation (7) with linear boundary conditions that come from the original problem. Due to the existence of the factor $\frac{1}{n}^m$, more chances for convergence may occur or even much faster convergence can be obtained better than the standard HAM. It should be noted that the case of $n=1$ in Equation (2), standard HAM can be reached. The q-HAM can be reformatted as follows:

We rewrite the nonlinear partial differential equation (1) in the form

$$Lu(x,t) + Au(x,t) + Bu(x,t) = 0$$

$$u(x,0) = f_0(x),$$

$$\frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = f_1(x), \tag{10}$$

$$\frac{\partial^{(z-1)} u(x,t)}{\partial^{(z-1)}}\Big|_{(t=0)} = f_{(z-1)}(x),$$

Where, $L = \frac{\partial^z}{(\partial t^z)}$, $z=1,2,\ldots$ is the highest partial derivative with respect to $t$, $A$ is a linear term, and $B$ is non-linear term. The so-called zero-order deformation Equation (2) becomes:

$$(1-nq)L\left[\varnothing(x,t;q) - u_0(x,t)\right] = qhH(x,t)(Lu(x,t) + Au(x,t) + Bu(x,t)) \tag{11}$$

we have the $m^{th}$ order deformation equation

$$L\left[u_m(x,t) - k_m u_{m-1}(x,t)\right] = hH(x,t)(Lu_{m-1}(x,t) + Au_{m-1}(x,t) + B(u^{-}_{m-1}(x,t))) \tag{12}$$

and hence

$$u_m(x,t) = k_m u_{m-1}(x,t) + hL^{-1}[H(x,t)(Lu_{m-1}(x,t) + Au_{m-1}(x,t) + B(u^{-}_{m-1}(x,t)))] \tag{13}$$

Now, the inverse operator $L^{-1}$ is an integral operator which is given by

$$L^{-1}(.) = \int\int\ldots\int(.)\underbrace{dt\,dt\ldots dt}_{z\ times} + c_1 t^{z-1} + c_2 t^{z-2} + \ldots + c_z \tag{14}$$

where $c_1, c_2, \ldots, c_z$ are integral constants.

To solve (10) by means of q-HAM, we choose the initial approximation:

$$u_0(x,t) = f_0(x) + f_1(x)t + f_2(x)\frac{t^2}{2!} + \ldots + f_{z-1}(x)\frac{t^{z-1}}{(z-1)!} \tag{15}$$

Let $(x, t)=1$, by means of Equations (14) and (15) then Equation (13) becomes

$$u_m(x,t) = k_m u_{m-1}(x,t) + h\int_0^t\int_0^t \ldots \int_0^t \left(\frac{\partial^z u_{m-1}(x,\tau)}{\partial \tau^z} + Au_{m-1}(x,\tau) + B(u^{-}_{m-1}(x,\tau))\right)\underbrace{d\tau\, d\tau \ldots d\tau}_{z\ times} \tag{16}$$

Now from times $\int_0^t\int_0^t \cdots \int_0^t \left(\frac{\partial^z u_{m-1}(x,\tau)}{\partial \tau^z}\right.\underbrace{d\tau\, d\tau \ldots d\tau}_{z\ times}$, we observe that there are repeated computations in each step which caused more consuming time. To cancel this, we use the following modification to (16):

$$u_m(x,t) = k_m u_{m-1}(x,t) + h\int_0^t\int_0^t \cdots \int_0^t \frac{\partial^z u_{m-1}(x,\tau)}{\partial \tau^z}\underbrace{d\tau\, d\tau \ldots d\tau}_{z\ times} + h$$

$$\int_0^t\int_0^t \cdots \int_0^t (Au_{m-1}(x,\tau) + B(u^-_{m-1}(x,\tau)))\underbrace{d\tau\, d\tau \ldots d\tau}_{z\ times}$$

$$= k_m u_{m-1}(x,t) + hu_{m-1}(x,t) - h\left(u_{m-1}(x,0) + t\frac{\partial u_{m-1}(x,0)}{\partial t} + \ldots + \frac{t^{z-1}}{(z-1)!}\frac{\partial^{z-1}u_{m-1}(x,0)}{\partial t^{z-1}}\right) + \quad (17)$$

$$+h\int_0^t\int_0^t \cdots \int_0^t (Au_{m-1}(x,\tau) + B(u^-_{m-1}(x,\tau)))\underbrace{d\tau\, d\tau \ldots d\tau}_{z\ times}$$

Now, for $m$=1, $k_m$=0 and

$$u_0(x,0) + t\frac{\partial u_0(x,0)}{\partial t} + \frac{t^2}{2!}\frac{\partial^2 u_0(x,0)}{\partial t^2}\ldots + \frac{t^{z-1}}{(z-1)!}\frac{\partial^{z-1}u_0(x,0)}{\partial t^{z-1}}$$

$$= f_0(x) + f_1(x)t + f_2(x)\frac{t^2}{2!} + \ldots + f_{z-1}(x)\frac{t^{z-1}}{(z-1)!} = u_0(x,t)$$

Substituting this equality into Equation (17), we obtain

$$u_1(x,t) = h\int_0^t\int_0^t \cdots \int_0^t (Au_0(x,\tau) + B(u_0(x,\tau)))\underbrace{d\tau\, d\tau \ldots d\tau}_{z\ times} \quad (18)$$

For $m$>1, $k_m$=$n$ and

$$u_m(x,0) = 0, \frac{\partial u_m(x,0)}{\partial t} = 0, \frac{\partial^2 u_m(x,0)}{\partial t^2} = 0, \ldots, \frac{\partial^{z-1}u_m(x,0)}{\partial t^{z-1}} = 0$$

Substituting this equality into Equation (17), we obtain

$$u_m(x,t) = (n+h)u_{m-1}(x,t) + h\int_0^t\int_0^t \cdots \int_0^t (Au_{m-1}(x,\tau) + B(u^-_{m-1}(x,\tau)))\underbrace{d\tau\, d\tau \ldots d\tau}_{z\ times} \quad (19)$$

We observe that the iteration in Equation (19) does not yield repeated terms and is also better than the iteration in Equation (16).

The standard q-HAM is powerful when $z$=1, and the series solution expression by q-HAM can be written in the form

$$u(x,t;n;h) \cong U_M(x,t;n;h) = \sum_{i=0}^{M} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i \quad (20)$$

However, when $z$≥2, there are too much additional terms where harder computations and more time consuming are performed. Hence, the closed form solution needs more number of iterations.

## THE PROPOSED MQ-HAM

When $z \geq 2$, we rewrite Equation (1) as the following system of the first-order differential equations

$$\begin{gathered} u_t = u_1 \\ u1_t = u2 \\ \vdots \\ u\{z-1\}_t = -Au(x,t) - Bu(x,t) \end{gathered} \tag{21}$$

Set the initial approximation

$$\begin{gathered} u_0\,(x,t) = f_0\,(x), \\ u1_0\,(x,t) = f_1\,(x), \\ \vdots \\ u\{z-1\}_0\,(x,t) = f(z-1)\,(x) \end{gathered} \tag{22}$$

Using the iteration formulas (18) and (19) as follows

$$\begin{gathered} u_1(x,t) = h\int_0^t \left(-u1_0\left(x,\tau\right)\right)d\tau, \\ u1_1(x,t) = h\int_0^t \left(-u2_0\left(x,\tau\right)\right)d\tau \\ \vdots \\ u\{z-1\}_1(x,t) = h\int_0^t \left(Au_0\left(x,\tau\right) + B(u_0\left(x,\tau\right))\right)d\tau \end{gathered} \tag{23}$$

For $m>1$, $k_m = n$ and

$$u_m\,(x,0)=0,\ u1_m\,(x,0)=0,\ u2_m\,(x,0)=0,\ldots,u\{z-1\}_m\,(x,0)=0$$

Substituting in Equation (17), we obtain

$$\begin{gathered} u_m\left(x,t\right) = \left(n+h\right)u_{m-1}\left(x,t\right) + h\int_0^t \left(-u1_{m-1}\left(x,\tau\right)\right)d\tau, \\ u1_m\left(x,t\right) = \left(n+h\right)u1_{m-1}\left(x,t\right) + h\int_0^t \left(-u2_{m-1}\left(x,\tau\right)\right)d\tau \\ \vdots \\ u\{z-1\}_m\left(x,t\right) = \left(n+h\right)u\{z-1\}_{m-1}\left(x,t\right) + h\int_0^t \left(Au_{m-1}\left(x,\tau\right) + B(u_{m-1}\left(x,\tau\right))\right)d\tau \end{gathered} \tag{24}$$

To illustrate the effectiveness of the proposed mq-HAM, comparison between mq-HAM and the standard q-HAM is illustrated by the following examples.

## ILLUSTRATIVE EXAMPLES[8,9]

We choose the following two cases when $z=2$ and $z=3$.

Case 1. $z=2$

Consider the modified Boussinesq equation

$$u_{tt} - u_{xxxx} - (u^3)_{xx} = 0 \tag{25}$$

subject to the initial conditions

$$u(x,0)=\sqrt{2}\text{sech}[x]$$

$$u_t(x,0)=\sqrt{2}\text{sech}[x]\tanh[x] \quad (26)$$

The exact solution is

$$u(x,t)=\sqrt{2}\text{sech}[x-t] \quad (27)$$

This problem solved by HAM (q-HAM [$n$=1]) and nHAM (mq-HAM [$n$=1]),[7] so we will solve it by q-HAM and mq-HAM and compare the results.

## IMPLEMENTATION OF Q-HAM

We choose the initial approximation

$$u_0(x,t)=u(x,0)+tu_t(x,0)$$
$$=\sqrt{2}\text{sech}[x]+t\sqrt{2}\text{sech}[x]\tanh[x] \quad (28)$$

and the linear operator:

$$L[\varnothing(x,t;q)]=\frac{\partial^2\varnothing(x,t;q)}{\partial t^2}, \quad (29)$$

with the property:

$$L[c_0+c1t]=0, \quad (30)$$

where $c_0$ and $c_1$ are real constants.
We define the nonlinear operator by

$$N[\varnothing(x,t;q)]=\frac{\partial^2\varnothing(x,t;q)}{\partial t^2}-\frac{\partial^4\varnothing(x,t;q)}{\partial x^4}-\frac{\partial^2}{\partial x^2}[\varnothing(x,t;q)]^3 \quad (31)$$

According to the zero-order deformation Equation (2) and the mth-order deformation equation (7) with

$$R(\vec{u}_{m-1})=\frac{\partial^2 u_{m-1}}{\partial t^2}-\frac{\partial^4 u_{m-1}}{\partial x^4}-\frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{m-1} u_{m-1-i}\sum_{j=0}^{i} u_j u_{i-j}\right) \quad (32)$$

The solution of the mth-order deformation equation (7) for $m\geq 1$ takes the form

$$u_m(x,t)=k_m u_{m-1}(x,t)+h\iint R(\vec{u}_{m-1})\,dt\,dt+c_0+c_1t \quad (33)$$

where the coefficients $c_0$ and $c_1$ are determined by the initial conditions:

$$u_m(x,0)=0, \qquad \frac{\partial u_m(x,0)}{\partial t}=0 \quad (34)$$

Obviously, we obtain

$$u_1(x,t)=-\frac{1}{960\sqrt{2}}ht^2\text{Sech}[x]^8(135(-5+56t^2)\text{Cosh}[x]-15(19+412t^2)\text{Cosh}[3x]-15\text{Cosh}[5x]+$$
$$540t^2\text{Cosh}[5x]+15\text{Cosh}[7x]-215t\text{Sinh}[x]+$$
$$6120t^3\text{Sinh}[x]-315t\text{Sinh}[3x]-1836t^3\text{Sinh}[3x]-95t\text{Sinh}[5x]+$$
$$108t^3\text{Sinh}[5x]+5t\text{Sinh}[7x])$$

$$u_2(x,t) = -\frac{1}{960\sqrt{2}} h(h+n)t^2\text{Sech}[x]^8 (135(-5+56t^2)\text{Cosh}[x]$$
$$-15(19+412t^2)\text{Cosh}[3x] - 15\text{Cosh}[5x] + 540t^2\text{Cosh}[5x] + 15\text{Cosh}[7x]$$
$$-215t\text{Sinh}[x] + 6120t^3\text{Sinh}[x] - 315t\text{Sinh}[3x] - 1836t^3\text{Sinh}[3x] \quad (34)$$
$$-95t\text{Sinh}[5x] + 108t^3\text{Sinh}[5x] + 5t\text{Sinh}[7x])$$
$$+h(-\frac{1}{160\sqrt{2}} ht\text{Sech}[x]^{10}(1+\text{Cosh}[2x]+\text{Sinh}[2x])^3 (1-6\text{Cosh}[2x]+\ldots$$

$u_m$ $(x, t)$, ($m$=3,4,…) can be calculated similarly. Then, the series solution expression by q-HAM can be written in the form:

$$u(x,t;n;h) \cong U_M(x,t;n;h) = \sum_{i=0}^{M} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i \quad (35)$$

Equation (35) is a family of approximation solutions to the problem (25) in terms of the convergence parameters h and $n$. To find the valid region of $h$, the $h$ curves given by the 3rd order q-HAM approximation at different values of $x$, $t$, and $n$ are drawn in Figures 1-3. This figure shows the interval of $h$ which the value of $U_3$ $(x, t; n)$ is constant at certain $x$, $t$, and n, We choose the line segment nearly parallel to the horizontal axis as a valid region of h which provides us with a simple way to adjust and control the convergence region. Figures 4 and 5 show the comparison between $U_3$ of q-HAM using different values of $n$ with the solution (27). The absolute errors of the 3rd order solutions q-HAM approximate using different values of n are shown in Figures 6 and 7.

## IMPLEMENTATION OF MQ-HAM

To solve Equation (25) by mq-HAM, we construct system of differential equations as follows
$u_t$ $(x, t)$=$v(x, t)$,

$$v_t(x,t) = \frac{\partial^4 u(x,t)}{\partial x^4} + \frac{\partial^2}{\partial x^2}[u(x,t)]^3 \quad (36)$$

with initial approximations

$$u_0(x,t) = \sqrt{2}\text{sech}[x], \qquad v_0(x,t) = \sqrt{2}\text{sech}[x]\tanh[x] \quad (37)$$

and the auxiliary linear operators

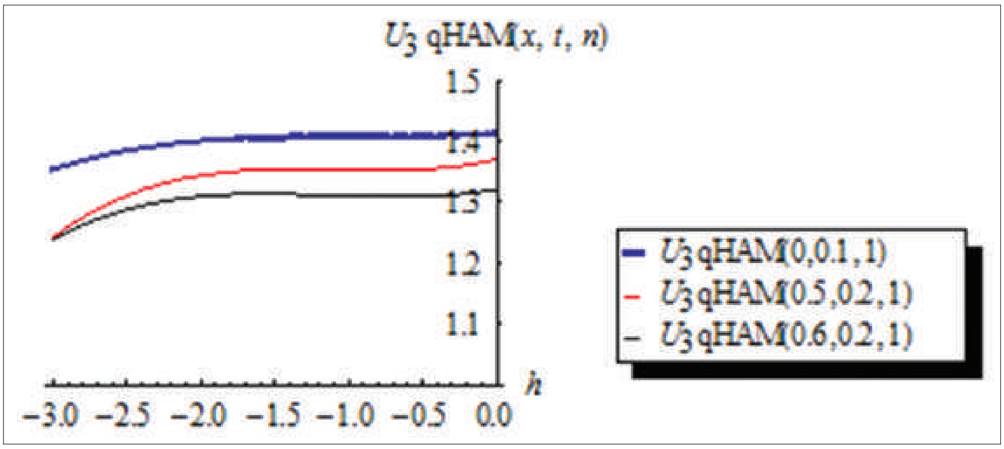


**Figure 1:** $h$ curve for the (q-HAM; $n$=1) (HAM) approximation solution $U_3$ $(x, t; 1)$ of problem (25) at different values of $x$ and $t$

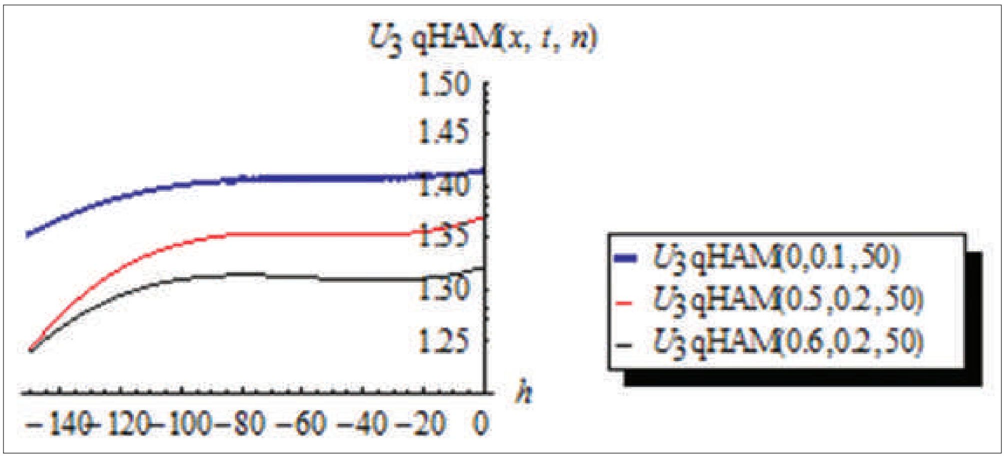


**Figure 2:** *h* curve for the (q-HAM; *n*=50) approximation solution $U_3$ (*x*, *t*; 50) of problem (25) at different values of *x* and *t*

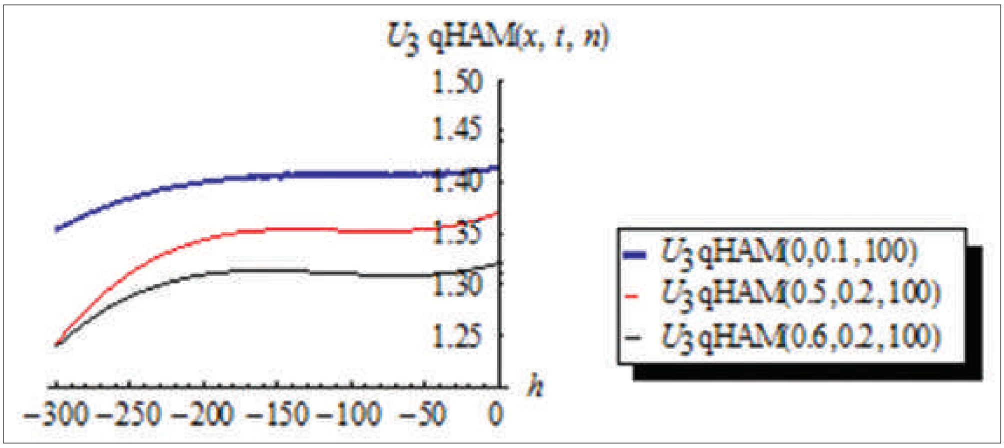


**Figure 3:** *h* curve for the (q-HAM; *n*=100) approximation solution $U_3$ (*x*, *t*; 100) of problem (25) at different values of *x* and *t*

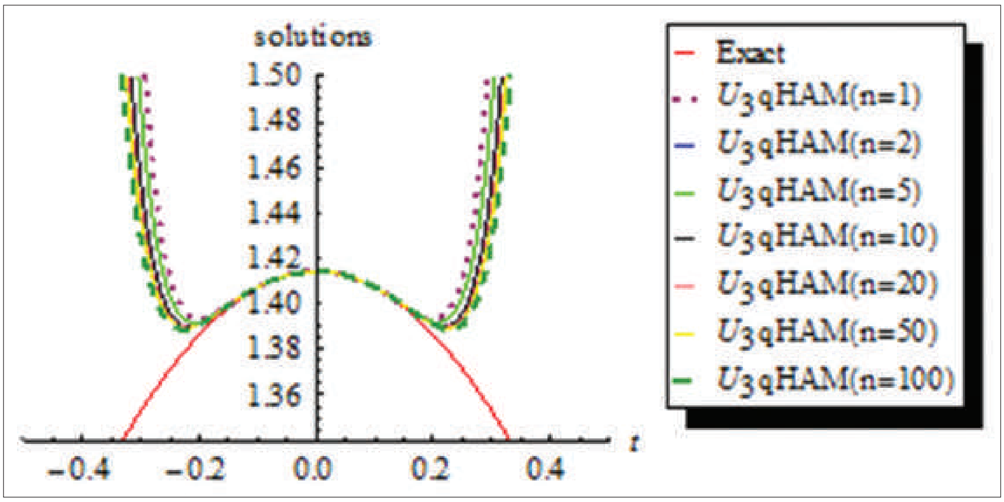


**Figure 4:** Comparison between $U_3$ of q-HAM (*n*=1, 2, 5, 10, 20, 50, 100) with exact solution of Equation (25) at *x*=0 with *h*=–1, *h*=–1.8, *h*=–4.5, (*h*=–8, *h*=–15.2, *h*=–37, *h*=–70), respectively

$$Lu\left(x,t\right)=\frac{\partial u(x,t)}{\partial t},\qquad Lv\left(x,t\right)=\frac{\partial v(x,t)}{\partial t} \tag{38}$$

and

$$Au_{m-1}\left(x,t\right)=-\frac{\partial^4 u_{m-1}(x,t)}{\partial x^4}$$

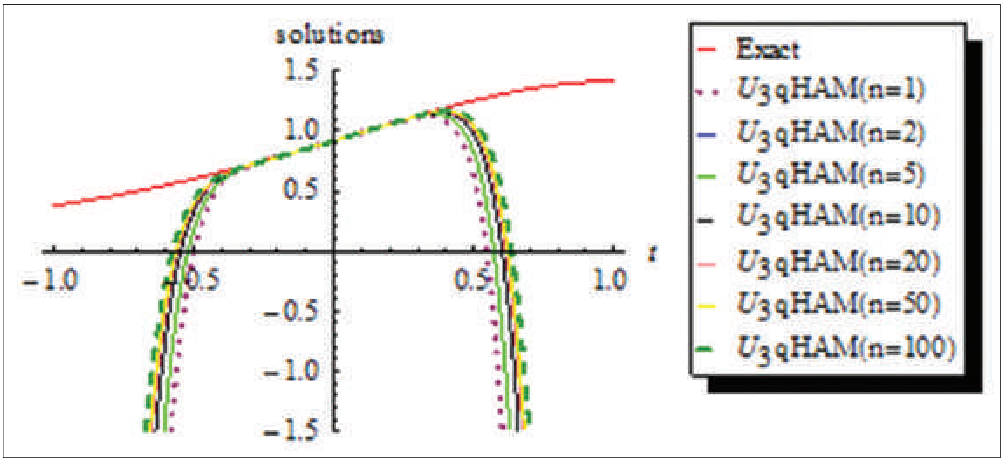


**Figure 5:** Comparison between $U_3$ of q-HAM ($n$=1, 2, 5, 10, 20, 50, 100) with exact solution of Equation (25) at $x$=1 with ($h$=–1, $h$=–1.8, $h$=–4.5, $h$=–8, $h$=–15.2, $h$=–37, $h$=–70), respectively

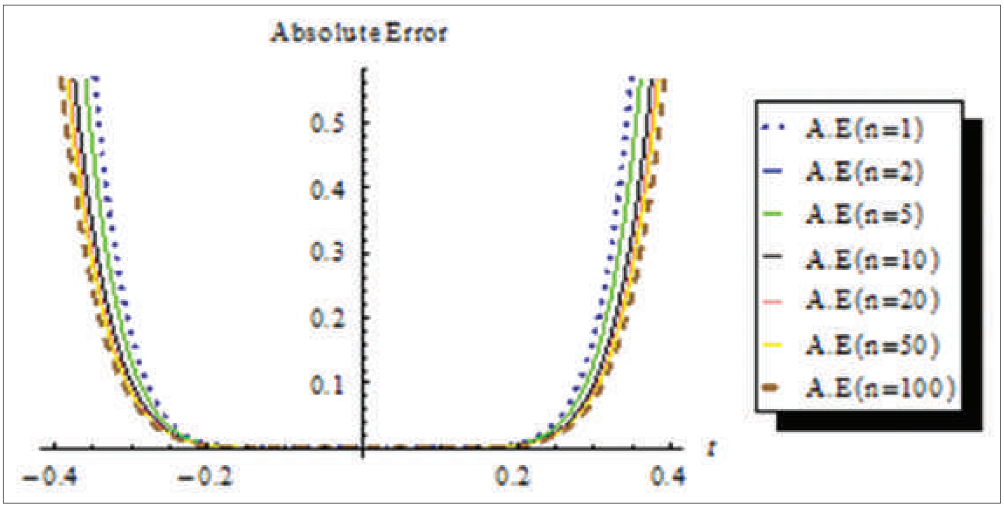


**Figure 6:** The absolute error of $U_3$ of q-HAM ($n$=1, 2, 5, 10, 20, 50, 100) for problem (25) at $x$=0 using ($h$=–1, $h$=–1.8, $h$=–4.5, $h$=–8, $h$=–15.2, $h$=–37, $h$=–70), respectively

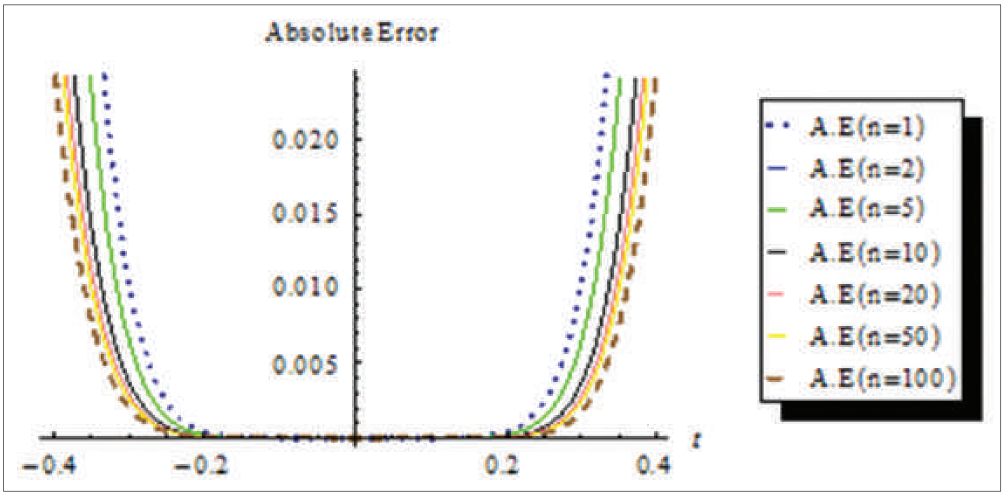


**Figure 7:** The absolute error of $U_3$ of q-HAM ($n$=1, 2, 5, 10, 20, 50, 100) for problem (25) at $x$=1 using ($h$=–1, $h$=–1.8, $h$=–4.5, $h$=–8, $h$=–15.2, $h$=–37, $h$=–70), respectively

$$Bu_{m-1}^{-}\left(x,t\right)=-\frac{\partial^2}{\partial x^2}\left(\sum\nolimits_{i=0}^{m-1} u_{m-1-i}(x,t)\sum\nolimits_{j=0}^{i} u_j\left(x,t\right)u_{i-j}(x,t)\right) \quad (39)$$

From Equations (23) and (24) we obtain:

$$u_1\left(x,t\right)=h\int_0^t \left(-v_0\left(x,\tau\right)\right)d\tau \quad (40)$$

$$v_1(x,t) = h\int_0^t \left(-\frac{\partial^4 u_0(x,\tau)}{\partial x^4} - \frac{\partial^2}{\partial x^2}\left(u_0(x,\tau)\right)^3\right)d\tau \,.$$

Now, form ≥2, we get

$$u_m(x,t) = (n+h)u_{m-1}(x,t) + h\int_0^t \left(-v_{m-1}(x,\tau)\right)d\tau \tag{41}$$

$$v_m(x,t) = (n+h)v_{m-1}(x,t) + h\int_0^t \left(-\frac{\partial^4 u_{m-1}(x,\tau)}{\partial x^4} - \frac{\partial^2}{\partial x^2}\left(\sum_{i=0}^{m-1} u_{m-1-i}(x,\tau)\sum_{j=0}^{i} u_j(x,\tau)u_{i-j}(x,\tau)\right)\right)d\tau$$

And the following results are obtained

$$u_1(x,t) = -\sqrt{2}ht\text{Sech}[x]\text{Tanh}[x]$$

$$v_1(x,t) = ht(\sqrt{2}\text{Sech}[x]^5 - \sqrt{2}\text{Sech}[x]\text{Tanh}[x]^4)$$

$$u_2(x,t) = \frac{h^2t^2(-3+\text{Cosh}[2x])\text{Sech}[x]^3}{2\sqrt{2}} - \sqrt{2}h(h+n)t\text{Sech}[x]\text{Tanh}[x]$$

$$v_2(x,t) = \frac{h^2t^2(-11+\text{Cosh}[2x])\text{Sech}[x]^3\text{Tanh}[x]}{2\sqrt{2}} + h(h+n)t(\sqrt{2}\text{Sech}[x]^5 - \sqrt{2}\text{Sech}[x]\text{Tanh}[x]^4)$$

$u_m(x, t)$, ($m$=3, 4,…) can be calculated similarly. Then, the series solution expression by mq- HAM can be written in the form:

$$u(x,t;n;h) \cong U_M(x,t;n;h) = \sum_{i=0}^{M} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i \tag{42}$$

Equation (42) is a family of approximation solutions to the problem (25) in terms of the convergence parameters h and $n$. To find the valid region of $h$, the $h$ curves given by the 3rd order mq-HAM approximation at different values of $x$, $t$, and $n$ are drawn in Figures 8-10. This figure shows the interval of $h$ which the value of $U_3(x, t; n)$ is constant at certain $x$, $t$, and $n$. We choose the line segment nearly parallel to the horizontal axis as a valid region of h which provides us with a simple way to adjust and control the convergence region. Figure 11 shows the comparison between $U_3$ of mq-HAM using different values of $n$ with the solution (27). The absolute errors of the 3th order solutions mq-HAM approximate using different values of n are shown in Figure 12. The results obtained by mq-HAM are more accurate than

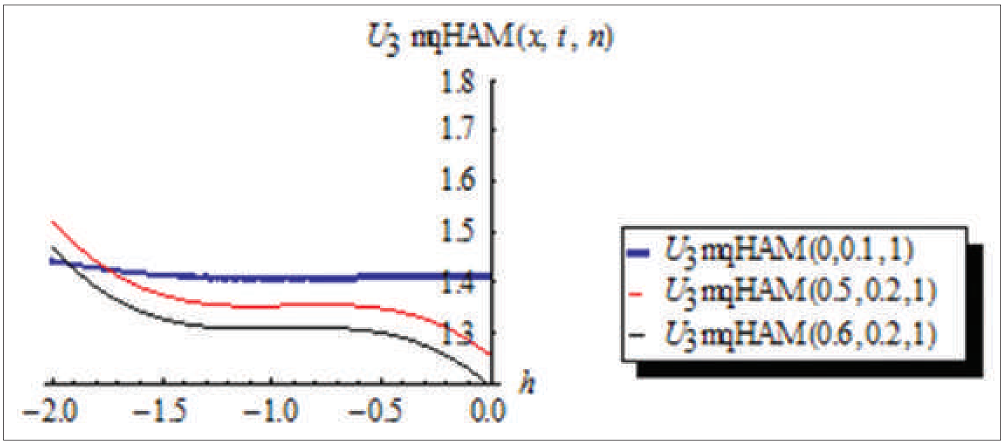


**Figure 8:** $h$ curve for the (mq-HAM; $n$=1) approximation solution $U_3(x, t; 1)$ of problem (25) at different values of $x$ and $t$

q-HAM at different values of $x$ and $n$, so the results indicate that the speed of convergence for mq-HAM with $n$>1 is faster in comparison with $n$=1 (nHAM). The results show that the convergence region of series solutions obtained by mq-HAM is increasing as $q$ is decreased, as shown in Figures 11 and 12. By increasing the number of iterations by mq-HAM, the series solution becomes more accurate, more efficient and the interval of $t$ (convergent region) increases, as shown in Figures 13-20.

Case 2. z=3

Consider the non-linear initial value problem:

$$u_{ttt}(x,t)+u_x(x,t)-2x\big(u(x,t)\big)^2+6\big(u(x,t)\big)^4=0 \tag{43}$$

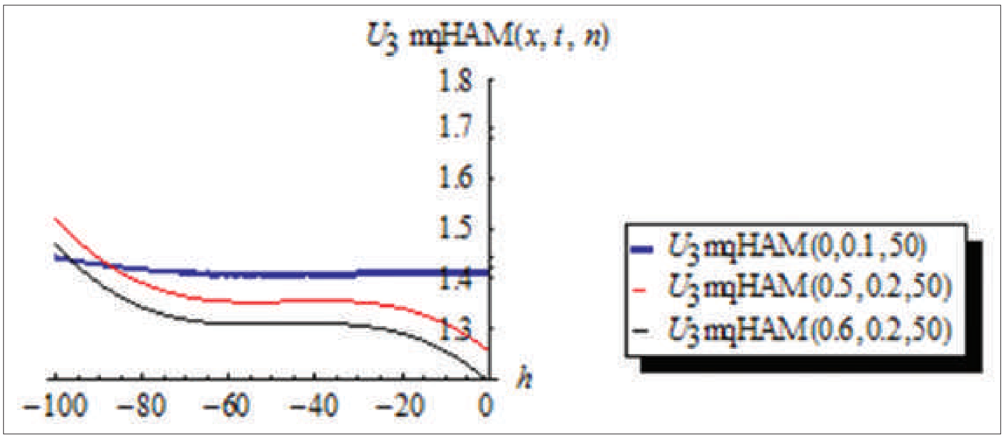


**Figure 9:** $h$ curve for the (mq-HAM; $n$=50) approximation solution $U_3$ ($x$, $t$; 50) of problem (25) at different values of $x$ and $t$

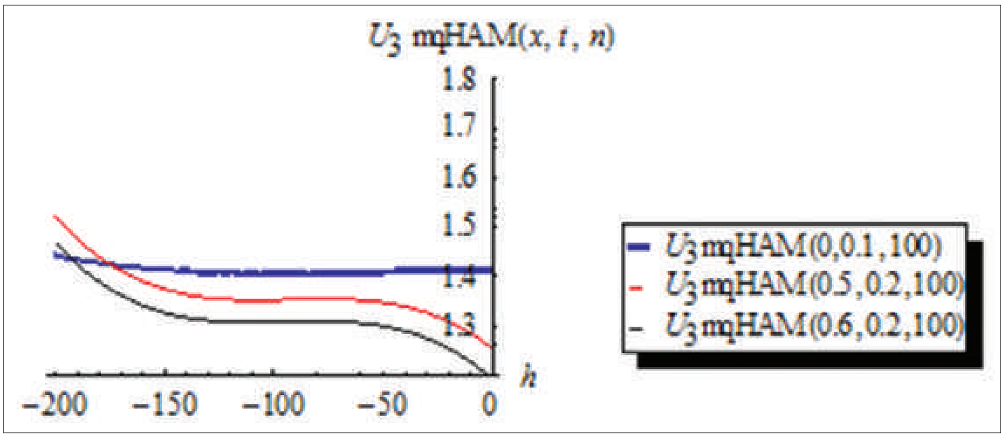


**Figure 10:** $h$ curve for the (mq-HAM; $n$=100) approximation solution $U_3$ ($x$, $t$; 100) of problem (25) at different values of $x$ and $t$

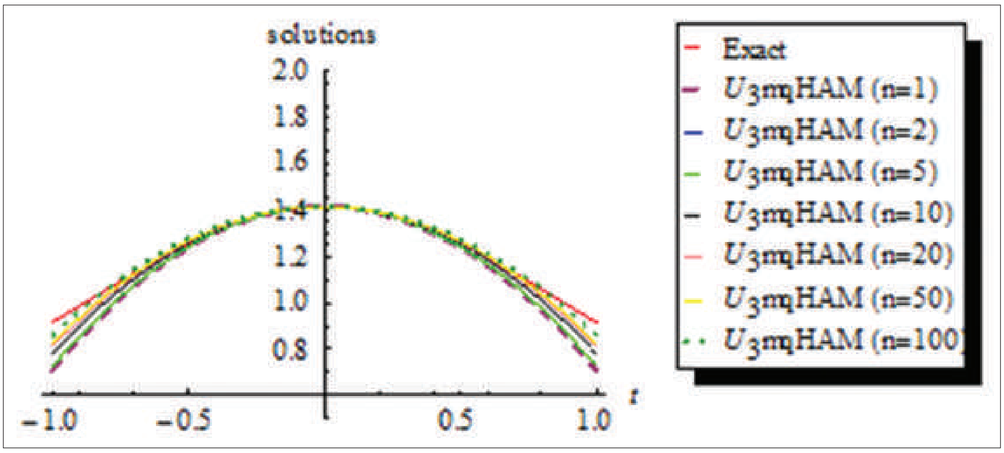


**Figure 11:** Comparison between $U_3$ ($x$, $t$) of mq-HAM ($n$=1, 2, 5, 10, 20, 50, 100) with exact solution of Equation (25) at $x$=0 with ($h$=–1, $h$=–1.8, $h$=–4.5, $h$=–8, $h$=–15.2, $h$=–37, $h$=–70), respectively

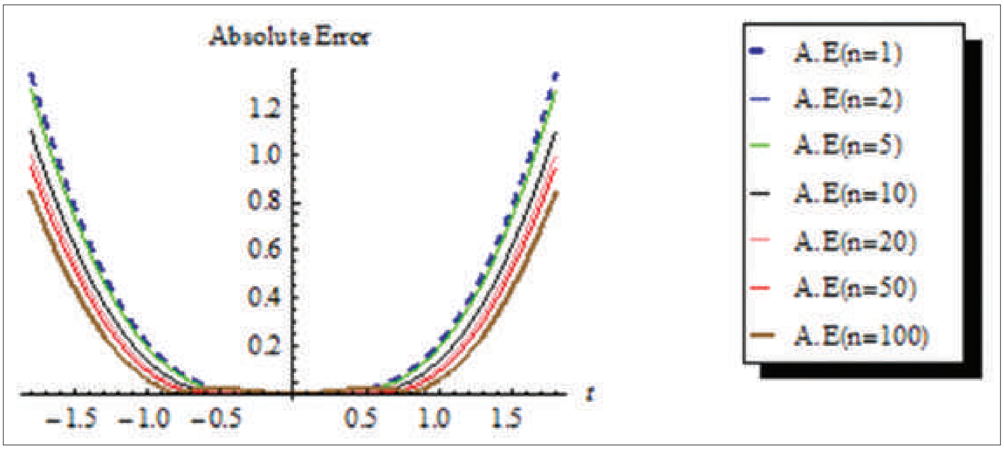


**Figure 12:** The absolute error of $U_3$ of mq-HAM ($n$=1, 2, 5, 10, 20, 50, 100) for problem (25) at $x$=0 using ($h$=–1, $h$=–1.8, $h$=–4.5, $h$=–8, $h$=–15.2, $h$=–37, $h$=–70), respectively

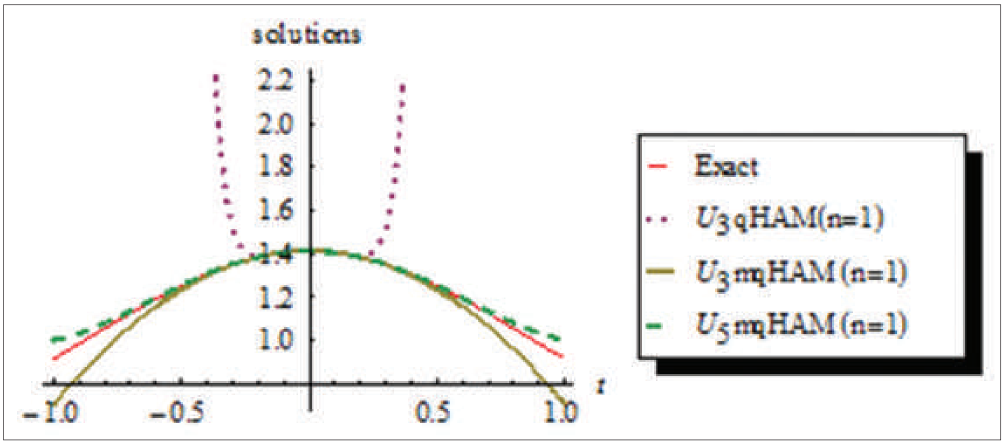


**Figure 13:** The comparison between the $U_3$ $(x, t)$ of q-HAM ($n$=1), $U_3$ $(x, t)$ of mq-HAM ($n$=1), $U_5$ $(x, t)$ of mq-HAM ($n$=1), and the exact solution of Equation (25) at $h$=–1 and $x$=0

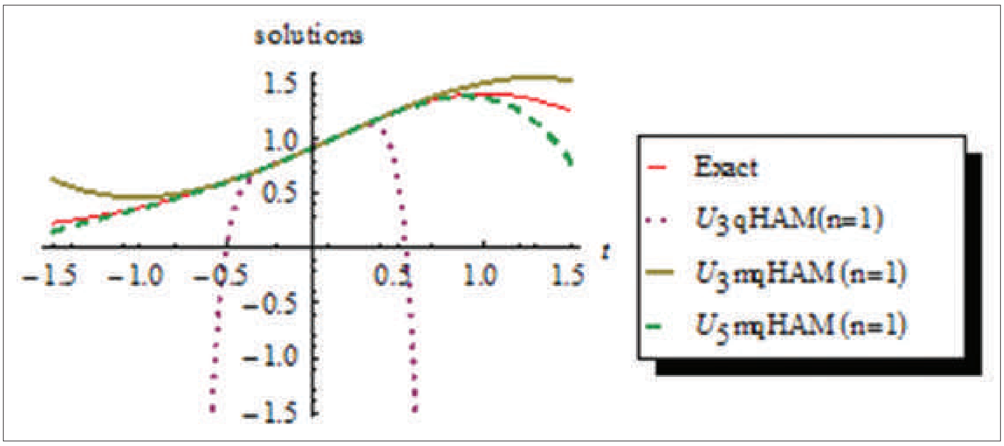


**Figure 14:** The comparison between the $U_3$ $(x, t)$ of q-HAM ($n$=1), $U_3$ $(x, t)$ of mq-HAM ($n$=1), $U_5$ $(x, t)$ of mq-HAM ($n$=1), and the exact solution of Equation (25) at $h$=–1 and $x$=1

Subject to the initial conditions

$$u(x,0) = -\frac{1}{x^2}, u_t(x,0) = -\frac{1}{x^4}, u_{tt}(x,0) = -\frac{2}{x^6} \tag{44}$$

The exact solution is

$$u(x,t) = \frac{1}{-x^2 + t} \tag{45}$$

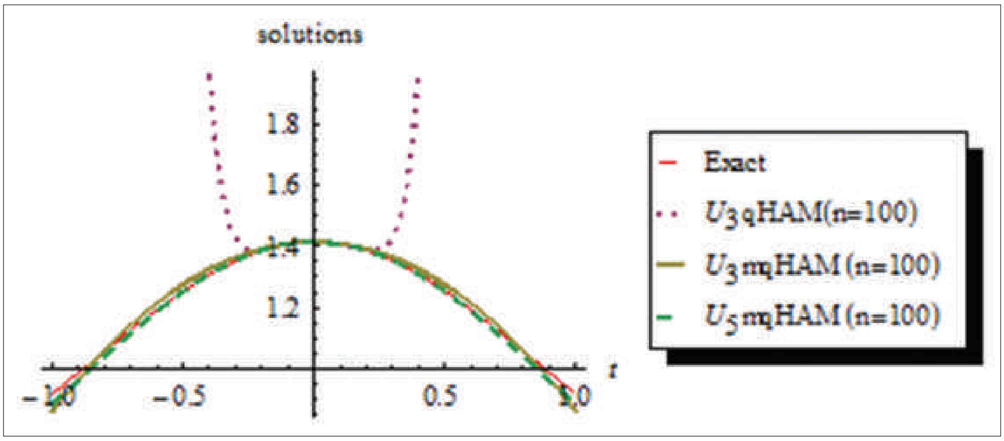


**Figure 15:** The comparison between the $U_3$ $(x, t)$ of q-HAM ($n$=100), $U_3$ $(x, t)$ of mq-HAM ($n$=100), $U_5$ $(x, t)$ of mq-HAM ($n$=100), and the exact solution of Equation (25) at $h$=–70 and $x$=0

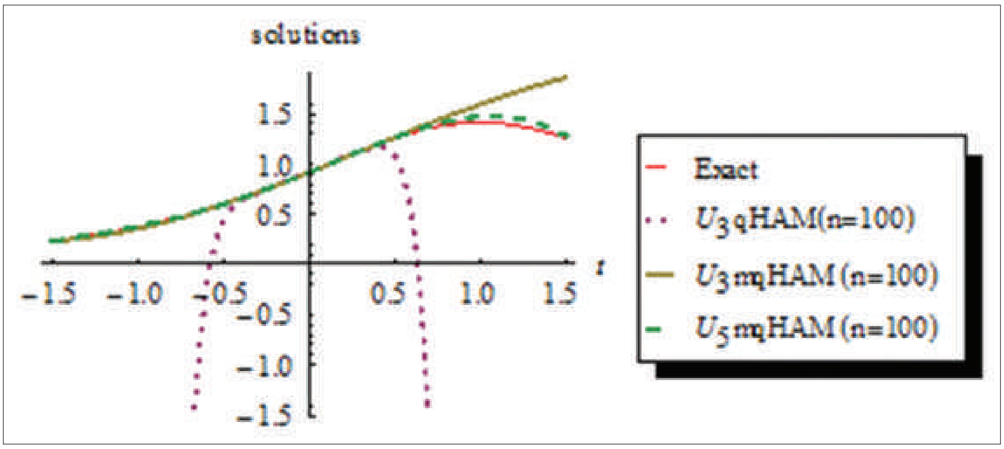


**Figure 16:** The comparison between the $U_3$ $(x, t)$ of q-HAM ($n$=100), $U_3$ $(x, t)$ of mq-HAM ($n$=100), $U_5$ $(x, t)$ of mq-HAM ($n$=100), and the exact solution of Equation (25) at $h$=–70 and $x$=1

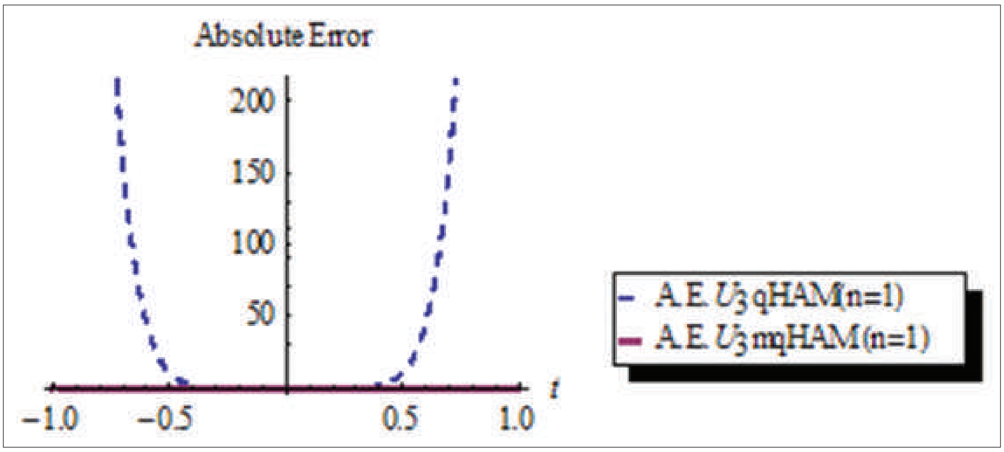


**Figure 17:** The comparison between the absolute error of $U_3$ $(x, t)$ of q-HAM ($n$=1) and $U_3$ $(x, t)$ of mq-HAM ($n$=1) of Equation (25) at $h$=–1, $x$=0 and-1≤$t$≤1

This problem solved by HAM (q-HAM ($n$=1)) and nHAM (mq-HAM ($n$=1)),[7] so we will solve it by q-HAM and mq-HAM and compare the results.

## IMPLEMENTATION OF Q-HAM

We choose the initial approximation

$$u_0\left(x,t\right)=-\frac{1}{x^2}-\frac{t}{x^4}-\frac{t^2}{x^6} \tag{46}$$

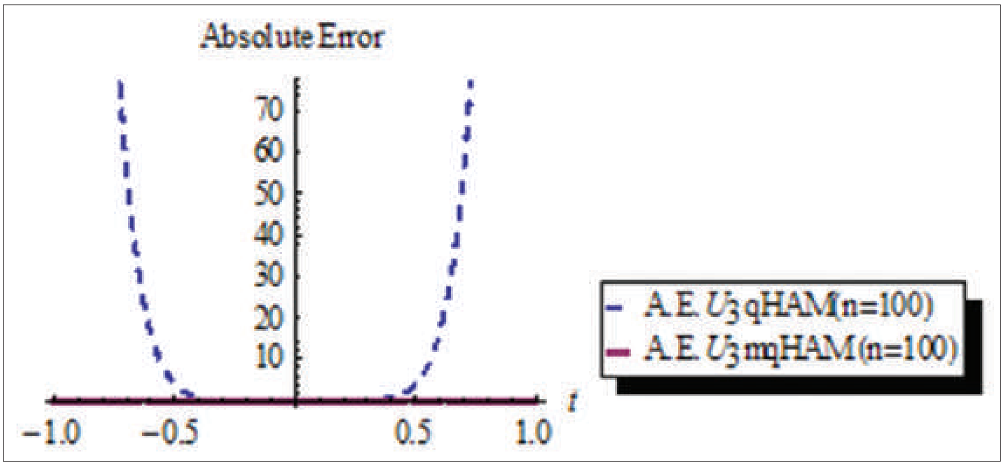


**Figure 18:** The comparison between the absolute error of $U_3$ $(x, t)$ of q-HAM ($n$=100) and $U_3$ $(x, t)$ of mq-HAM ($n$=100) of Equation (25) at $h$=–70, $x$=0 and $-1 \le t \le 1$

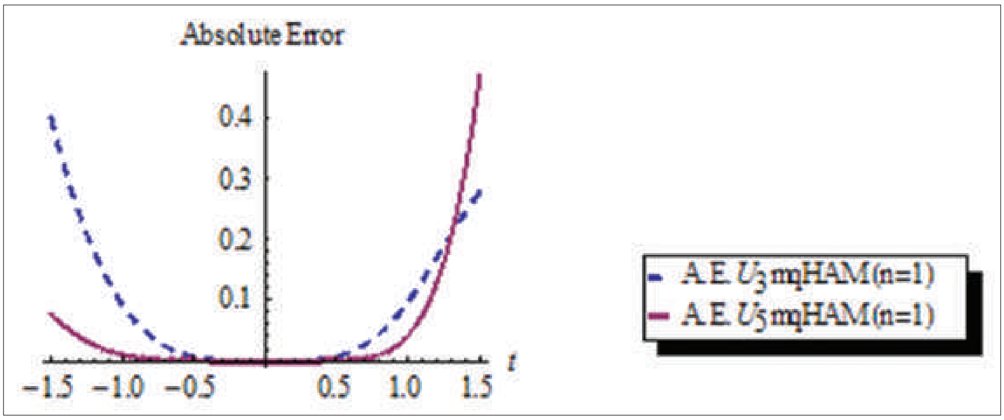


**Figure 19:** The comparison between the absolute error of $U_3$ $(x, t)$ of mq-HAM ($n$=1) and $U_5$ $(x, t)$ of mq-HAM ($n$=1) of Equation (25) at $h$=–1, $x$=1 and $-1.5 \le t \le 1.5$

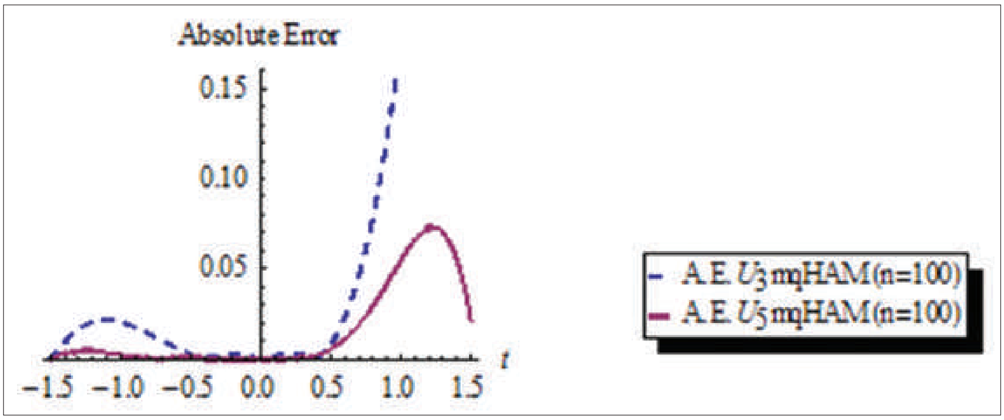


**Figure 20:** The comparison between the absolute error of $U_3$ $(x, t)$ of mq-HAM ($n$=100) and $U_5$ $(x, t)$ of mq-HAM ($n$=100) of Equation (25) at $h$=–70, $x$=1 and $-1.5 \le t \le 1.5$

and the linear operator:

$$L[\varnothing(x,t;q)] = \frac{\partial^3 \varnothing(x,t;q)}{\partial t^3} \quad (47)$$

with the property:

$$L\left[c_0 + c_1 t + c_2 t^2\right] = 0 \quad (48)$$

where $c_0$, $c_1$, and $c_2$ are real constants.

Next, we define the nonlinear operator by

$$N\left[\varnothing\left(x,t;q\right)\right]=\frac{\partial^3\varnothing(x,t;q)}{\partial t^3}+\frac{\partial\varnothing(x,t;q)}{\partial x}-2x[\varnothing\left(x,t;q\right)]^2+6[\varnothing\left(x,t;q\right)]^4 \tag{49}$$

According to the zero-order deformation Equation (2) and the $m^{th}$-order deformation equation (7) with

$$R\left(\vec{u}_{m-1}\right)=\frac{\partial^3 u_{m-1}}{\partial t^3}+\frac{\partial u_{m-1}}{\partial x}-2x\sum_{i=0}^{m-1} u_i u_{m-1-i}+6\sum_{i=0}^{m-1} u_{m-1-i}\sum_{j=0}^{i} u_{i-j}\sum_{k=0}^{j} u_k u_{j-k} \tag{50}$$

The solution of the $m^{th}$-order deformation equation (7) for $m\geq 1$ becomes:

$$u_m\left(x,t\right)=k_m u_{m-1}\left(x,t\right)+h\iiint R\left(\vec{u}_{m-1}\right)dt\,dt\,dt+c_0+c_1t+c_2t^2 \tag{51}$$

where the coefficients $c_0$, $c_1$ and $c_2$ are determined by the initial conditions:

$$u_m\left(x,0\right)=0,\qquad \frac{\partial u_m\left(x,0\right)}{\partial t}=0,\frac{\partial^2 u_m\left(x,0\right)}{\partial t^2}=0 \tag{52}$$

We now successively obtain:

$$u_1\left(x,t\right)=\frac{1}{2310x^{24}}ht^3(14t^8+77t^7x^2+275t^6x^4+660t^5x^6$$
$$+2310t^2x^{12}+2310tx^{14}+2310x^{16}-22t^4x^8(-57+x^5)-77t^3x^{10}(-24+x^5))$$

$$u_2\left(x,t\right)=\frac{1}{2310x^{24}}hnt^3(14t^8+77t^7x^2+275t^6x^4+660t^5x^6+$$
$$2310t^2x^{12}+2310tx^{14}+2310x^{16}-22t^4x^8(-57+x^5)-77t^3x^{10}(-24+x^5))$$
$$-\frac{1}{24443218800x^{42}}h^2t^3(519792t^{17}+5197920t^{16}x^2+30603300t^{15}x^4+$$
$$127288980t^{14}x^6+10475665200t^5x^{24}-\ldots$$

$u_m$ $(x, t)$, $(m$=3,4,…) can be calculated similarly. Then, the series solution expression by q- HAM can be written in the form:

$$u\left(x,t;n;h\right)\cong U_M\left(x,t;n;h\right)=\sum_{i=0}^{M} u_i\left(x,t;n;h\right)\left(\frac{1}{n}\right)^i \tag{53}$$

Equation (53) is a family of approximation solutions to the problem (43) in terms of the convergence parameters h and $n$. To find the valid region of $h$, the $h$ curves given by the 5th order q-HAM approximation at different values of $x$, $t$, and $n$ are drawn in Figures 21-23. This figure shows the interval of $h$ which the value of $U_5$ $(x, t; n)$ is constant at certain $x$, $t$ and $n$. We choose the line segment nearly parallel to the horizontal axis as a valid region of h which provides us with a simple way to adjust and control the convergence region. Figure 24 shows the comparison between $U_5$ of q-HAM using different values of $n$ with the solution 45. The absolute errors of the 5th order solutions q-HAM approximate using different values of n are shown in Figure 25.

## IMPLEMENTATION OF MQ-HAM

To solve Equation (43) by mq-HAM, we construct system of differential equations as follows

$$u_t\,(x,t)=v(x,t),$$
$$v_t\,(x,t)=w(x,t) \tag{54}$$

With initial approximations

$$u_0\left(x,t\right)=-\frac{1}{x^2},\qquad v_0\left(x,t\right)=-\frac{1}{x^4},\qquad w_0\left(x,t\right)=-\frac{2}{x^6} \tag{55}$$

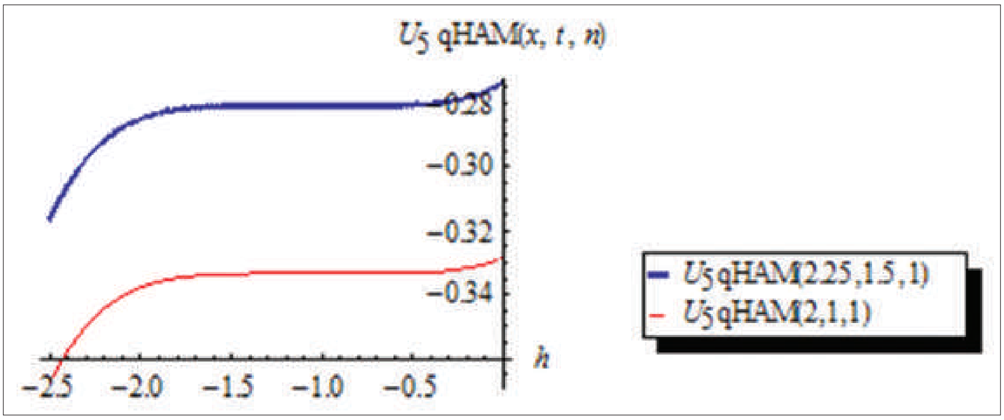


**Figure 21:** *h* curve for the (q-HAM; *n*=1) (HAM) approximation solution $U_5$ (*x*, *t*; 1) of problem (43) at different values of *x* and *t*

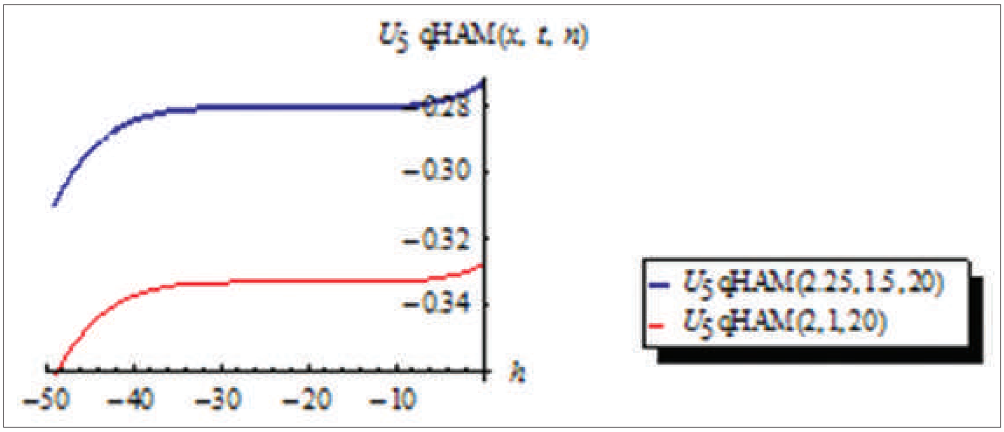


**Figure 22:** *h* curve for the (q-HAM; *n*=20) approximation solution $U_5$ (*x*, *t*; 20) of problem (43) at different values of *x* and *t*

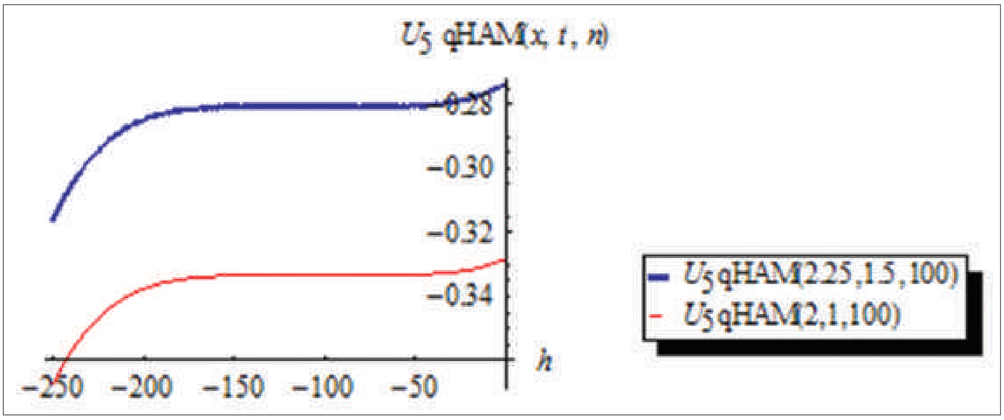


**Figure 23:** *h* curve for the (q-HAM; *n*=100) approximation solution $U_5$ (*x*, *t*; 100) of problem (43) at different values of *x* and *t*

And the auxiliary linear operators

$$Lu(x,t)=\frac{\partial u(x,t)}{\partial t},\ Lv(x,t)=\frac{\partial v(x,t)}{\partial t},\ Lw(x,t)=\frac{\partial w(x,t)}{\partial t} \tag{56}$$

And

$$Au_{m-1}(x,t)=\frac{\partial u_{m-1}(x,t)}{\partial x}$$

$$Bu_{m-1}(x,t)=-2x\sum_{i=0}^{m-1} u_i u_{m-1-i}+6\sum_{i=0}^{m-1} u_{m-1-i}\sum_{j=0}^{i} u_{i-j}\sum_{k=0}^{j} u_k u_{j-k} \tag{57}$$

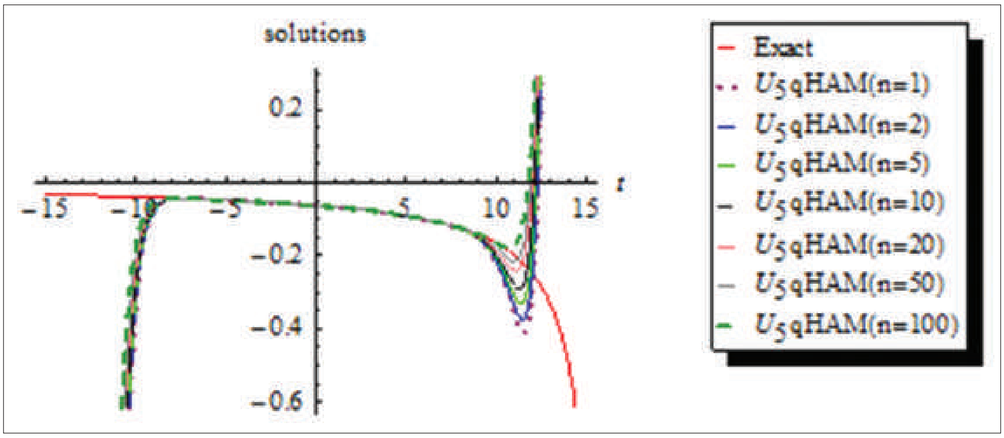


**Figure 24:** Comparison between $U_5$ of q-HAM ($n$=1, 2, 5, 10, 20, 50, 100) with exact solution of Equation (43) at $x$=4 with ($h$=–1, $h$=–1.97, $h$=–4.83, $h$=–8.45, $h$=–18.3, $h$=–44.75, $h$=–86), respectively

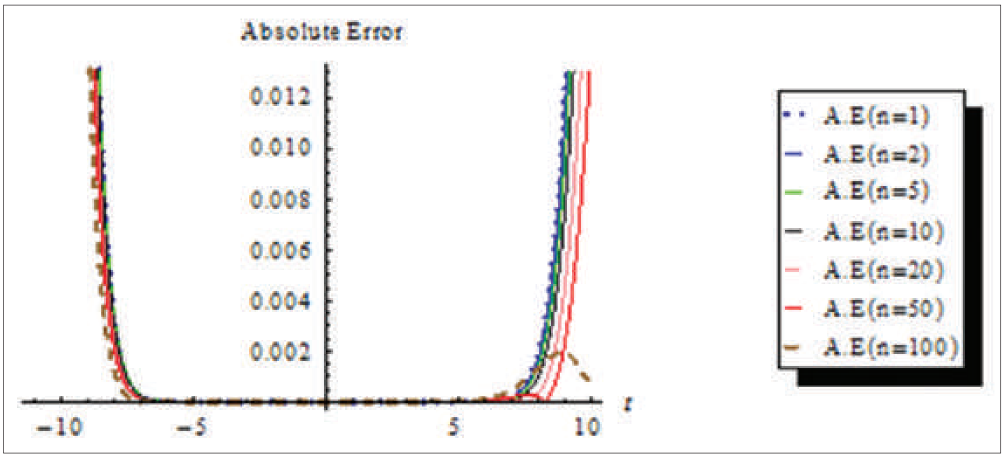


**Figure 25:** The absolute error of $U_5$ of q-HAM ($n$=1, 2, 5, 10, 20, 50, 100) for problem (43) at $x$=4 using $h$=–1, $h$=–1.97, $h$=–4.83, $h$=–8.45, $h$=–18.3, $h$=–44.75, $h$=–86), respectively

From Equations (23) and (24) we obtain

$$u_1(x,t)=h\int_0^t \left(-v_0(x,\tau)\right)d\tau$$

$$v_1(x,t)=h\int_0^t \left(-w_0(x,\tau)\right)d\tau \tag{58}$$

$$w_1(x,t)=h\int_0^t\left(\frac{\partial u_0(x,\tau)}{\partial x}-2x\left(u_0(x,\tau)\right)^2+6\left(u_0(x,\tau)\right)^4\right)d\tau$$

For $m\geq 2$,

$$u_m(x,t)=(n+h)u_{m-1}(x,t)+h\int_0^t\left(-v_{m-1}(x,\tau)\right)d\tau$$

$$v_m(x,t)=(n+h)v_{m-1}(x,t)+h\int_0^t\left(-w_{m-1}(x,\tau)\right)d\tau \tag{59}$$

$$w_m(x,t)=(n+h)w_{m-1}(x,t)+h\int_0^t$$
$$\left(\frac{\partial u_{m-1}(x,t)}{\partial x}-2x\sum_{i=0}^{m-1}u_i u_{m-1-i}+6\sum_{i=0}^{m-1}u_{m-1-i}\sum_{j=0}^{i}u_{i-j}\sum_{k=0}^{j}u_k u_{j-k}\right)d\tau$$

The following results are obtained

$$u_1(x,t)=\frac{ht}{x^4}$$

$$u_2(x,t)=-\frac{h^2t^2}{x^6}+\frac{h(h+n)t}{x^4}$$

$$u_3(x,t)=h(\frac{h^2t^3}{x^8}-\frac{h^2t^2}{x^6}-\frac{hnt^2}{x^6})+(h+n)(-\frac{h^2t^2}{x^6}+\frac{h(h+n)t}{x^4})$$

$u_m$ (*x*, *t*), (*m*=4, 5,…) can be calculated similarly. Then, the series solution expression by mq-HAM can be written in the form:

$$u(x,t;n;h)\cong U_M(x,t;n;h)=\sum_{i=0}^{M} u_i(x,t;n;h)\left(\frac{1}{n}\right)^i \quad (60)$$

Equation (60) is a family of approximation solutions to the problem (43) in terms of the convergence parameters h and *n*. To find the valid region of h, the h curves given by the 5th order mq-HAM approximation at different values of *x*, *t*, and *n* are drawn in Figures 26-28. This figure shows the interval of *h* which the value of $U_5$ (*x*, *t*; *n*) is constant at certain *x*, *t*, and *n*. We choose the line segment nearly parallel to the horizontal axis as a valid region of h which provides us with a simple way to adjust and control the convergence region. Figure 29 shows the comparison between $U_5$ of mq-HAM using different values of *n* with the solution (45). The absolute errors of the 5th order solutions mq-HAM approximate using different values of n are shown in Figure 30. The results obtained by mq-HAM are more accurate than q-HAM at different values of *x* and *n*, so the results indicate that the speed of convergence for mq-HAM with *n*>1 is faster in comparison with *n*=1. (nHAM). The results show that the convergence region of series solutions obtained by mq-HAM is increasing as *q* is decreased, as shown in Figures 29-36.

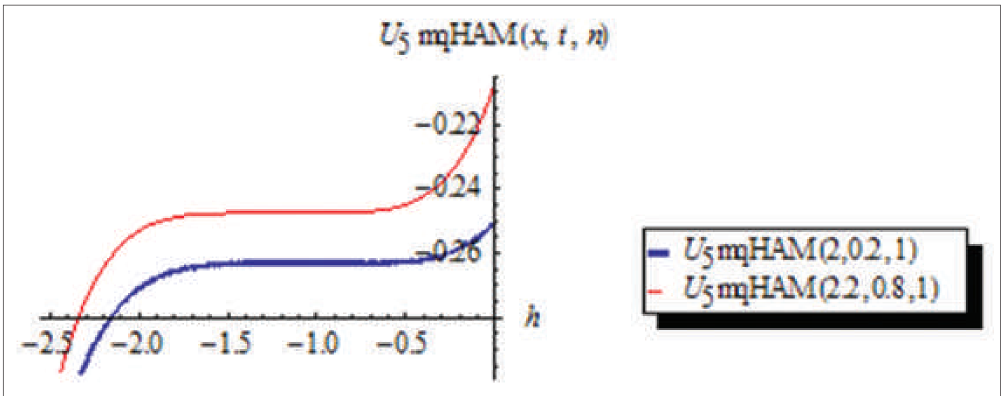


**Figure 26:** *h* curve for the (mq-HAM; *n*=1) approximation solution $U_5$ (*x*, *t*; 1) of problem (43) at different values of *x* and *t*

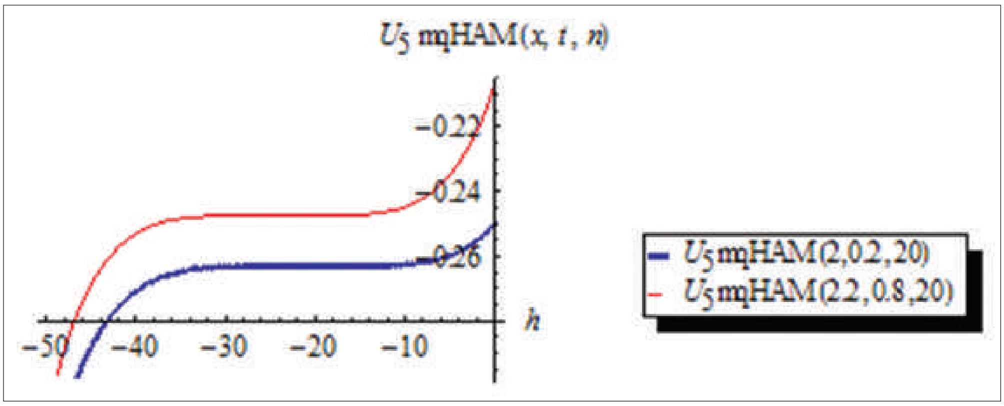


**Figure 27:** *h* curve for the (mq-HAM; *n*=20) approximation solution $U_5$ (*x*, *t*; 20) of problem (43) at different values of *x* and *t*

By increasing the number of iterations by mq-HAM, the series solution becomes more accurate, more efficient and the interval of $t$ (convergent region) increases, as shown in Figures 31-36.

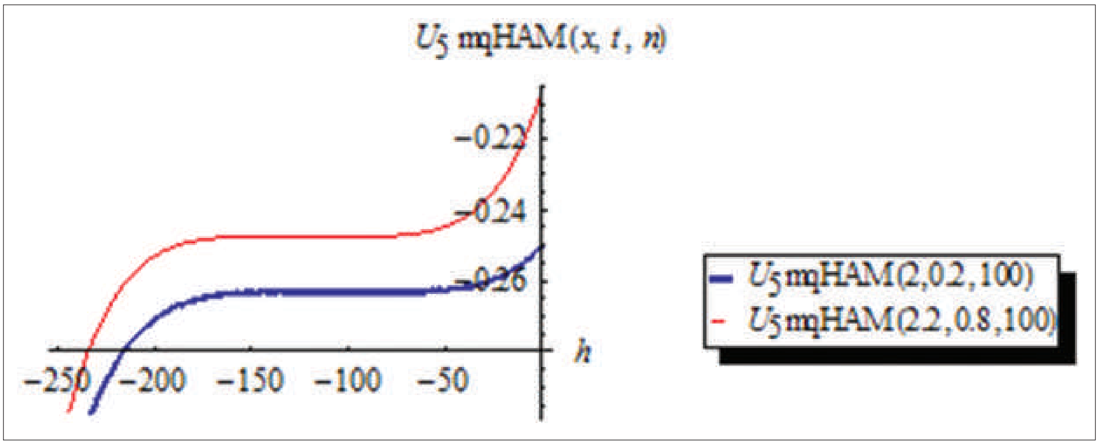


**Figure 28:** $h$ curve for the (mq-HAM; $n$=100) approximation solution $U_5$ ($x$, $t$; 100) of problem (43) at different values of $x$ and $t$

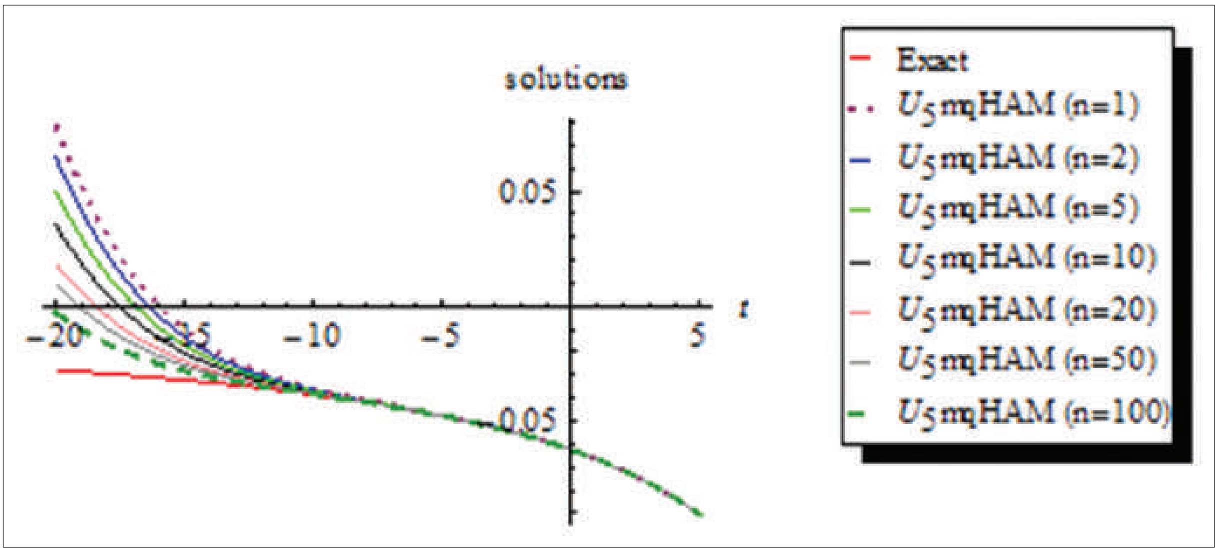


**Figure 29:** Comparison between $U_5$ of mq-HAM ($n$=1, 2, 5, 10, 20, 50, 100) with exact solution of Equation (43) at $x$=4 with ($h$=–1, $h$=–1.97, $h$=–4.83, $h$=–9.45, $h$=–18.3, $h$=–44.75, $h$=–86), respectively

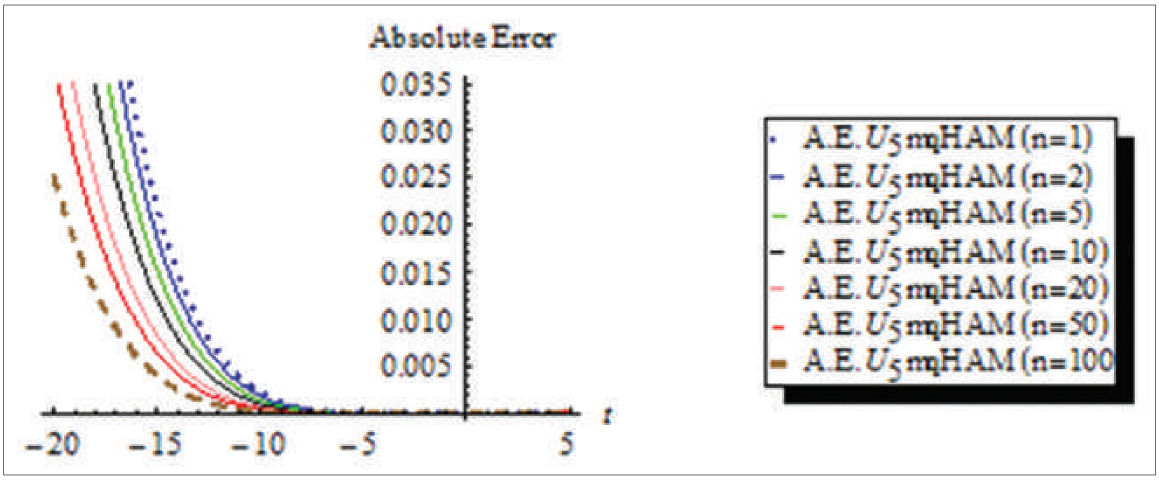


**Figure 30:** The absolute error of $U_5$ of mq-HAM ($n$=1, 2, 5, 10, 20, 50, 100) for problem (43) at $x$=4, $-20 \le t \le 5$ using $h$=–1, $h$=–1.97, $h$=–4.83, $h$=–9.45, $h$=–18.3, $h$=–44.75, $h$=–86), respectively

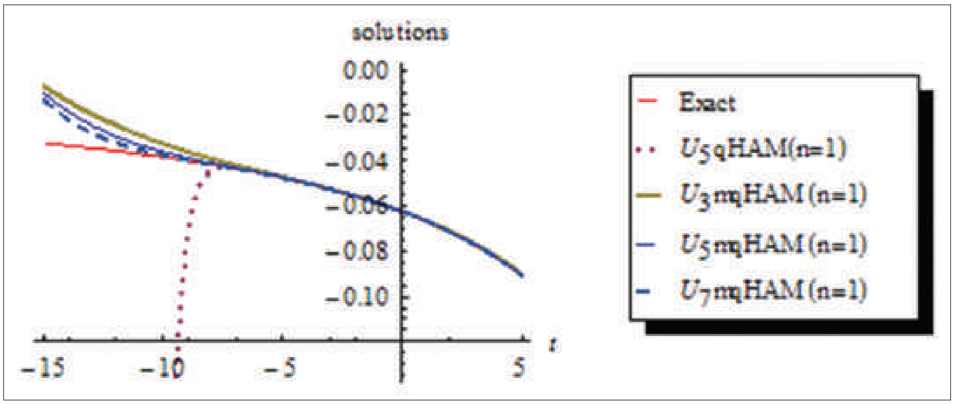


**Figure 31:** The comparison between the $U_5$ $(x, t)$ of q-HAM ($n$=1), $U_3$ $(x, t)$ of mq-HAM ($n$=1), $U_5$ $(x, t)$ of mq-HAM ($n$=1), $U_7$ $(x, t)$ of mq-HAM ($n$=1), and the exact solution of Equation (43) at $h$=–1 and $x$=4

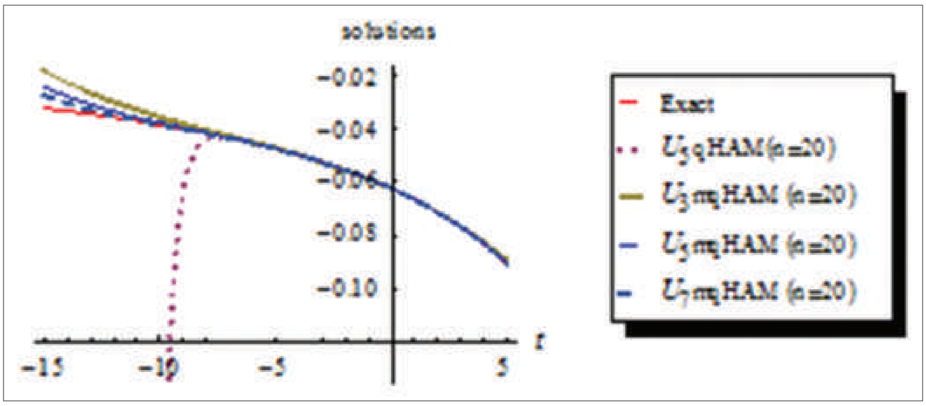


**Figure 32:** The comparison between the $U_5$ $(x, t)$ of q-HAM ($n$=20), $U_3$ $(x, t)$ of mq-HAM ($n$=20), $U_5$ $(x, t)$ of mq-HAM ($n$=20), $U_7$ $(x, t)$ of mq-HAM ($n$=20), and the exact solution of Equation (43) at $h$=–18.3 and $x$=4

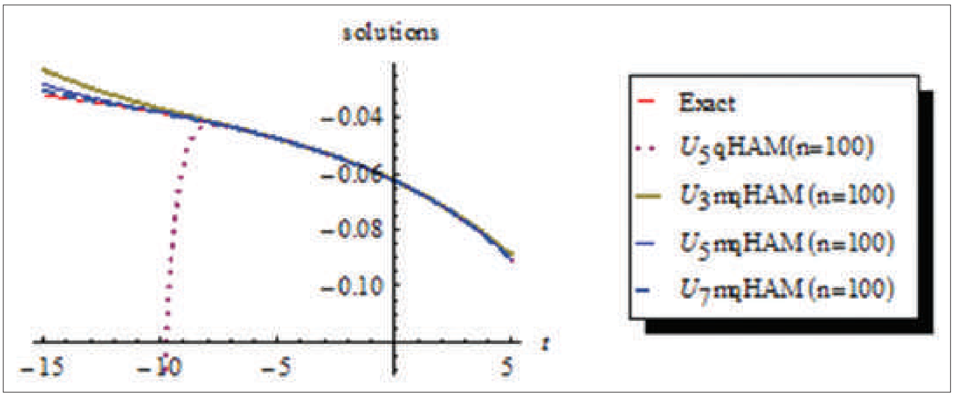


**Figure 33:** The comparison between the $U_5$ $(x, t)$ of q-HAM ($n$=100), $U_3$ $(x, t)$ of mq-HAM ($n$=100), $U_5$ $(x, t)$ of mq-HAM ($n$=100), $U_7$ $(x, t)$ of mq-HAM ($n$=100), and the exact solution of (43) at $h$=–86 and $x$=4

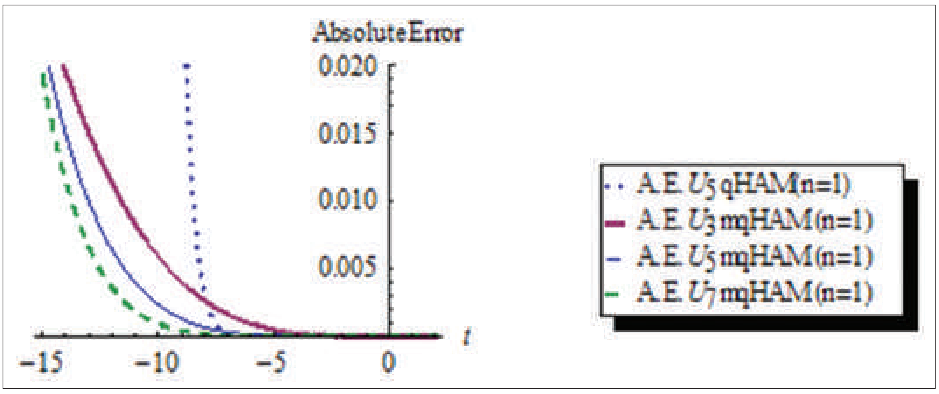


**Figure 34:** The comparison between the absolute error of $U_5$ $(x, t)$ of q-HAM ($n$=1), $U_3$ $(x, t)$ of mq-HAM ($n$=1), $U_5$ $(x, t)$ of mq-HAM ($n$=1), and $U_7$ $(x, t)$ of mq-HAM ($n$=1) of Equation (43) at $h$=–1, $x$=4 and $-15 \le t \le 2$

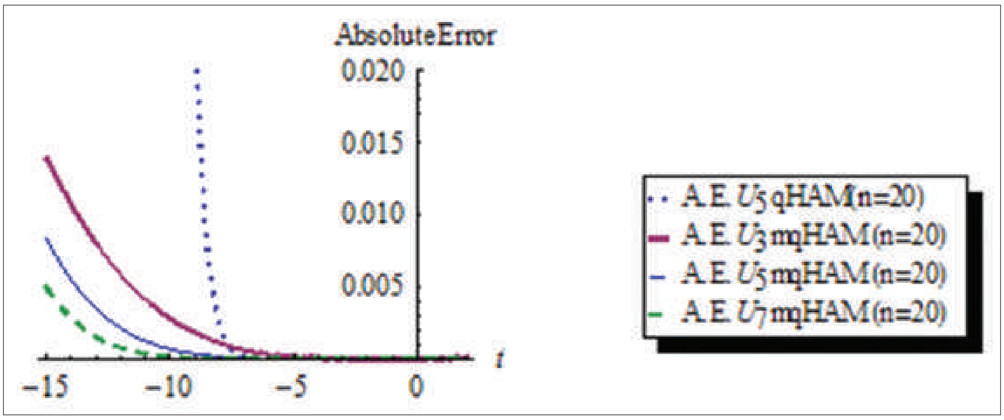


**Figure 35:** The comparison between the absolute error of $U_5$ (*x*, *t*) of q-HAM (*n*=20), $U_3$ (*x*, *t*) of mq-HAM (*n*=20), $U_5$ (*x*, *t*) of mq-HAM (*n*=20), and $U_7$ (*x*, *t*) of mq-HAM (*n*=20) of Equation (43) at $h$=–18.3, $x$=4 and $-15 \le t \le 2$

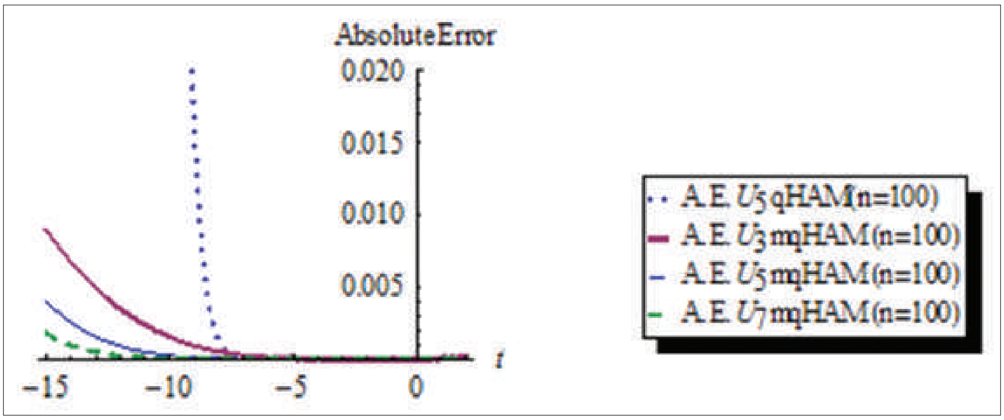


**Figure 36:** The comparison between the absolute error of $U_5$ (*x*, *t*) of q-HAM (*n*=100), $U_3$ (*x*, *t*) of mq-HAM (*n*=100), $U_5$ (*x*, *t*) of mq-HAM (*n*=100), and $U_7$ (*x*, *t*) of mq-HAM (*n*=100) of Equation (43) at $h$=–86, $x$=4 and $-15 \le t \le 2$

## CONCLUSION

A mq-HAM was proposed. This method provides an approximate solution by rewriting the nth-order non-linear differential equation in the form of *n* first-order differential equations. The solution of these *n* differential equations is obtained as a power series solution. It was shown from the illustrative examples that the mq-HAM improves the performance of q-HAM and nHAM.

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