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On The Inverse Function Theorem and its Generalization in the Unitary Space

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ABSTRACT

It is obvious that the inverse function theorem holds in the Banach space for R. In my paper on the generalized inverse function theorem, it was observed that the inverse function theorem also holds for R^n. However, in this paper, I attempted to establish that it holds in the unitary space and consequently can be extended to C^n; the generalized unitary space.

Key words: Norm space, continuity, differentiability, inverse function theorem

THE INVERSE FUNCTION THEOREM IN R

A function F could fail to be one to one but may be so on a subset S of D_F and by this we mean that $F(X_1)$ and $F(X_2)$ are distinct, whenever X_1 and X_2 are distinct points of S. Hence, F is not invertible but when F_S is defined on S by $F_s(X)=F(X), X\in S$, and left undefined for $X\notin S$ then F_s is invertible. We say that F_s is the restriction of F to S and that F_s^{-1} is the inverse of F restricted to S. The domain of F_s^{-1} is $F(S)$. If F is one to one on a neighborhood of X_0, we say that F is locally invertible on X_0 and if this true for every X_0 in a set S, we say that F is locally invertible on S.

Definition 1.1: [Riez [8]], [Williams[10]] A function $F:R^n\to R^n$ is regular on an open set S if F is one to one and continuously differentiable on S and $JF(X)\neq 0$, if $X\in S$. Also we may say that F is regular on an arbitrary set S if F is regular on an open set containing S.

Theorem 1.1: [Athanassius[1]], [Erwin[6]] Suppose that $F:R^n\to R^n$ is regular on an open set S, and let $G=F_s^{-1}$ then $F(S)$ is open, G is continuously differentiable on $F(S)$ and $G'(U)=F'(X)^{-1}$, where $U=F(X)$.

Moreover, since G is one to one on $F(S)$, G is regular on $F(S)$.

Definition 1.2: If F is regular on an open set S, we say that F_s^{-1} is a branch of F^{-1}. Hence, it is possible to better define a branch of F^{-1} on a set $T\subset R(F)$ if and only if $T=F(S)$ where F is regular on S. Note that any subset of $R(F)$ that does not have this property cannot have a branch of F^{-1} defined on them.

Theorem 1.2 (the inverse function theorem) [Athanassius[1]], [Erwin[6]]: Let $F:R^n\to R^n$ be continuously differentiable on an open set S and suppose that $JF(X)\neq 0$ on S. Then, if $X_0\in S$, there is an open neighborhood N of X_0 on which F is regular. Moreover, $F(N)$ is open and $G=F_N^{-1}$ is continuously differentiable on $F(N)$ with $G'(U)=[F'(X)]^{-1}$ (where $U=F(X), U\in F(N)$).

Corollary 1.3: If F is continuously differentiable on a neighborhood of X_0 and $JF(X_0)\neq 0$, then there is an open neighborhood N of X_0 on which the conclusion of theorem 1.2 holds.

THE INVERSE FUNCTION THEOREM ON THE UNITARY SPACE

Here, we discuss the inverse function theorem in a plane other than the reals and in precise the

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unitary space C^n. As preliminary in this section, we introduce the following concepts.

Local invertibility

A complex function F is one to one only on a subset S of D_F where D_F is complex points. This in general may fail but that the assertion holds means that $F(Z_1)$ and $F(Z_2)$ are distinct, whenever Z_1 and Z_2 are distinct points of S so that F is not invertible except if F_s is defined on S by $F_s(Z)=F(Z), Z \in S,$

Then, F_s is invertible. On the other hand, F_s is the restriction of F to S and F_s^{-1} is the inverse of F restricted to S and the domain of F_s^{-1} is $F(S)$. If F is one to one on a neighborhood of Z_0, we say that F is locally invertible at Z_0. If this is true for every Z_0 in a set S, then F is locally invertible on S.

Regular invertible functions

Definition 2.2.1: A complex function $F: C^n \to C^n$ is regular on an open set S and let $G=F_s^{-1}$. Then, $F(S)$ is open, G is continuously differentiable on $F(S)$ and $G(U)=(F(z))^{-1}$, where $U=F(Z)$. Moreover, since G is one to one on $F(S)$, G is regular on $F(S)$.

Definition 2.2.2: We say that F_s^{-1} is a branch of F^{-1} if F is regular on an open set S. More so, this definition implies that F_s^{-1} is a branch of F^{-1} on a set $T \subset C(F)$ if and only if $T=F(S)$, where F is regular on S. Note that any open subset of $C(F)$ that does not have this property cannot be said to have a branch defined on it.

Theorem 2.2 (the inverse function theorem): Let $F: C^n \to C^n$ be continuously differentiable on an open set S and suppose that $JF(Z) \neq 0$ on S. Then, if $Z_0 \in S$, then there is an open neighborhood N of Z_0 on which F is regular. More so, $F(N)$ is open and $G=F_N^{-1}$ is continuously differentiable on $F(N)$, with $G'(N)=\left[F'(z)\right]^{-1}$ (where $U=F(Z)$), $U \in F(N)$.

Corollary 2.2.3: If F is continuously differentiable on a neighborhood of Z_0 and $JF(Z_0) \neq 0$, then there is an open neighborhood N of Z_0 on which the conclusion of theorem 2.2 holds.

GENERALIZED INVERSE FUNCTION THEOREM IN THE UNITARY SPACE

Generalized local invertibility

A set of complex functions F_i are/is one to one only on a subset S of D_{F_i} where D_{F_i} is complex points. This in general may fail but that the assertion holds mean that $F_i(z_1)$ and $F_i(z_2)$ are distinct points of S so that $F_i's$ is not invertible except F_{i_s} is defined on S by $F_{i_s}(z_i)=F_i(z_i)$, $z_i \in S$ and left undefined for $z_i \in S$ and then F_{i_s} is invertible.

On the other hand, F_{i_s} is restrictions of F_i to S and $F_{i_s}^{-1}$ is the inverses of $F_i's$ restricted to S and the domain of $F_{i_s}^{-1}$ is $F(S)$. If $F_i's$ is one to one z_0 neighborhoods, we say that $F_i's$ is locally invertible each at z_0. If this is true for every z_0 in a set S, then $F_i's$ is locally invertible on S.

Generalized regular invertible functions

Definition 3.2.1: Complex functions $F_i: C^n \to C^n$ are each regular on an open set S and $J_iF_i(z_i) \neq 0$ if $z_i \in S$. We also say that F_i^s is each regular on an arbitrary set S if $F\_i^{'}s$ is regular on an open set containing S.

Theorem 3.2.1. Suppose that $F_i: C^n \to C^n$ are regular on an open set S and if $G_i=F_{i_s}^{-1}$, then $F_i(S)$ is open and $G_i's$ is continuously differentiable on $F_i(S)$ while $G_i(U)=(F_i(z_i))^{-1}$, where $U_i=F_i(z_i)$. Moreover, since $G_i's$ is one to one on $F_i(S)$, $G_i's$ irregular on $F_i(S)$.\\

Definition 3.2: We say that $F_{i_s}^{-1}$ is branches of F_i^{-1} if F_i is regular on an open set S. More so, this definition implies that $F_i's$ is branches of F_i^{-1} on a $T_i \subset R(F_i)$ if and only if $T_i=F_i(S)$, where $F_i's$ is regular on S. Note that any open subsets of $R(F_i)$ that do not have this property cannot be said to have branches defined on them.

MAIN RESULTS

Theorem 3.2 [the generalized inverse function theorem in the unitary space]

Let $F_i : C^n \to C^n$ be a set of continuously differentiable functions on an open set S. Suppose that each $J_i F_i(z_i) \neq 0$ on S. Then, if $z_i \in S$, there are open neighborhoods N_i of z_i on which $F_i's$ is regular. More so, $F_i(N_i)$ is each open with

$$F(N) = \bigcup_{j=1}^{n}\{F_i(N_i)\}$$

and

$$G = \bigcup_{i=1}^{n}\{G_i\} = \bigcup_{i=1}^{n}\{F_{i_{N_i}}\} = F_N^{-1}$$

Continuously differentiable on $\bigcup_{i=1}^{n}\{F_i(N_i)\}$ such that $G'(N) = \bigcup_{i=1}^{n}\{G_i(N_i)\} = \left[\bigcup_{i=1}^{n}\{F_i(z_i)\}\right]^{-1}$.

where $\bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} F_i(z_i), \bigcup_{i=1}^{n} U_i \in \bigcup_{i=1}^{n} F_i(N_i)$.

Proof: First, we show that if $X_0 \in S$, then a neighborhood of $\bigcup_{i=1}^{n} F_i(X_0)$ is in $\bigcup_{i=1}^{n} F_i(S)$. This implies that $\bigcup_{i=1}^{n} F_i(S)$ is open.

Since S is open, there is a $\bigcup_{i=1}^{n}\rho_i > 0$ such that $\bigcup_{i=1}^{n} B_{i_{\rho_i}}(X_0) \subset S$. Let $\bigcup_{i=1}^{n} B_i$ be the boundary of $\bigcup_{i=1}^{n} B_{i_{\rho_i}}(X_0)$, thus

$$B = \bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n}\{X_i\}\bigcup_{i=1}^{n} X_i - X_0 = \bigcup_{i=1}^{n} p_i = p \tag{3.1}$$

The functions

$$\sigma = \bigcup_{i=1}^{n}\sigma_i(X_i) = \bigcup_{i=1}^{n} F_i(X_i) - F_i(X_0)$$

are continuous on S and therefore on $\bigcup_{i=1}^{n} B_i$ which is compact. Hence, there is a point $\bigcup_{i=1}^{n} X_i$ in $\bigcup_{i=1}^{n} B_i$ where $\bigcup_{i=1}^{n}\sigma_i(X_i)$ attain its minimum value say, $\bigcup_{i=1}^{n} m_i$ on $\bigcup_{i=1}^{n} B_i$ Moreover, $\bigcup_{i=1}^{n} m_i > 0$ since $\bigcup_{i=1}^{n} Z_i \neq 0$ each $\bigcup_{i=1}^{n} F_i$ is one to one on S.

Therefore, $\bigcup_{i=1}^{n} F(Z_i) - F(Z_0) \geq \bigcup_{i=1}^{n} m_i > 0$ if

$$\bigcup_{i=1}^{n} Z_i - Z_0 = \bigcup_{i=1}^{n}\rho_i \tag{3.2}$$

The set

$$\left\{U_i U_i - F_i(Z_0) \leq \bigcup_{i=1}^{n}\frac{m_i}{2}\right\}$$

is a neighborhood of $\bigcup_{i=1}^{n} F_i(Z_0)$.

We will show that it is a subset of $\bigcup_{i=1}^{n} F_i(S)$. To see this, let $\bigcup_{i=1}^{n} U_i$ be a set of fixed points in this set. Thus,

$$\bigcup_{i=1}^{n} U_i - F_i(Z_i) < \bigcup_{i=1}^{n}\frac{m_i}{2} \tag{3.3}$$

Consider the function

$$\bigcup_{i=1}^{n}\sigma_i(Z_i) = \bigcup_{i=1}^{n} U_i - F_i(Z_i)^2$$

which is continuous on S. Note that $\bigcup_{i=1}^{n}\sigma_i \geq \bigcup_{i=1}^{n}\frac{m_i}{4}$ if

$$\bigcup_{i=1}^{n} Z_i - Z_0 = \bigcup_{i=1}^{n}\rho_i$$

Since if $\bigcup_{i=1}^{n} Z_i - Z_0 = \bigcup_{i=1}^{n}\rho_i$, then

$$\bigcup_{i=1}^{n} U_i - F_i(Z_i) = \bigcup_{i=1}^{n}\left(U_i - F_i(Z_0)\right) + \begin{pmatrix} F_i(Z_0) \\ -F_i(Z_i) \end{pmatrix}$$

$$\geq \bigcup_{i=1}^{n} F_i(X_0) - F_i(X_i) - \bigcup_{i=1}^{n} U_i - F_i(X_0) \geq$$

$$\bigcup_{i=1}^{n}\left(m_i - \frac{m_i}{2}\right) = \bigcup_{i=1}^{n}\frac{m_i}{2}$$

that is, from Equations (3.2) and (3.3).

Since $\bigcup_{i=1}^{n}\sigma_i$ is continuous on S, $\bigcup_{i=1}^{n}\sigma_i$ attains a minimum value μ on the compact set $\overline{B_\rho(Z_0)}$ that is there are $\overline{Z_i}$ in $\overline{B_\rho(Z_0)}$ such that

$$\bigcup_{i=1}^{n}\sigma_i(Z_i) \geq \bigcup_{i=1}^{n}\sigma_i(\overline{Z_i}) = \mu, \bigcup_{i=1}^{n} Z_i \in \overline{B_\rho(Z_0)} \tag{3.4}$$

Setting

$$\bigcup_{i=1}^{n} Z_i = Z_0,$$

We conclude from Equation (3.3) that

$$\bigcup_{i=1}^{n} \sigma_i(\bar{Z}) = \mu \le \bigcup_{i=1}^{n} \sigma_i(Z_0) < \bigcup_{i=1}^{n} \frac{m_i}{4}$$

Because of Equations (3.1) and (3.4), this rules out the possibility that $\bigcup_{i=1}^{n} Z_i \in B$, so $\bigcup_{i=1}^{n} \overline{Z_i} \in B_\rho(Z_0)$.

Now, we want to show that $\mu = 0$; that is

$$\bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} F_i(\overline{Z_i})$$

To this end, we note that $\bigcup_{i=1}^{n} \sigma_i(Z_i)$ can be written as

$$\bigcup_{i=1}^{n} \sigma_i(Z_i) = \sum_{i=1}^{n} (U_{i,j} - f_{i,j}(Z_i))^2$$

So $\bigcup_{i=1}^{n} \sigma_i$ is differentiable on $B_\rho(Z_0)$. Therefore, the first partial derivatives of $\bigcup_{i=1}^{n} \sigma_i$ are all zero at the local minimum point $\bigcup_{i=1}^{n} \overline{Z_i}$, so

$$\sum_{i=1}^{n} \frac{\partial f_{i.j}(\bar{Z})}{\partial x_{i,j}} \left(U_{i,j} - f_{i,j}(\bar{Z})\right) = 0 \text{for} 1 \le i \le n$$

or in matrix form

$$\bigcup_{i=1}^{n} F_i'(\overline{Z_i})\left(U_i - F_i(\overline{Z_i})\right) = 0$$

Since $\bigcup_{i=1}^{n} F_i'(Z_i)$ is non-singular, this implies that

$$\bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} F_i(\overline{Z_i})$$

Thus, we have shown that every U that satisfies (3.3) is in $\bigcup_{i=1}^{n} F_i(S)$ is open.

Next, we show that $\bigcup_{i=1}^{n} G_i$ is continuous on $\bigcup_{i=1}^{n} F_i(S)$ and Z_0 is the unique point in S such that $\bigcup_{i=1}^{n} F_i(Z_0) = U_0$. Since $\bigcup_{i=1}^{n} F_i'(Z_0)$ is invertible, there exists $\lambda_i > 0$ and an open neighborhood $\bigcup_{i=1}^{n} N$ of Z_0 such that $\bigcup_{i=1}^{n} N \subset S$ and

$$\bigcup_{i=1}^{n} F_i(Z_i) - F_i(Z_0) \ge \bigcup_{i=1}^{n} \lambda_i Z_i - Z_0 \text{ if} \bigcup_{i=1}^{n} Z_i \in \bigcup_{i=1}^{n} N_i \tag{3.5}$$

Since $\bigcup_{i=1}^{n} F_i$ satisfies the hypothesis of the present theorem on $\bigcup_{i=1}^{n} N_i$, the first part of this proof shows that $\bigcup_{i=1}^{n} F_i(N_i)$ is an open set containing $U_i = \bigcup_{i=1}^{n} F_i(Z_0)$. Therefore, there is a $\delta > 0$ such that $\bigcup_{i=1}^{n} Z_i = \bigcup_{i=1}^{n} G_i(U_i)$ is in $\bigcup_{i=1}^{n} N_i$ if $\bigcup_{i=1}^{n} U_i \in B_\delta(U_0)$. Setting $\bigcup_{i=1}^{n} Z_i = \bigcup_{i=1}^{n} G_i(U_i)$ and $Z_0 = \bigcup_{i=1}^{n} G_i(U_0)$ in Equation (3.5), yields

$$\bigcup_{i=1}^{n} F_i(G_i(U_i)) - F_i(G_i(U_0)) \ge \bigcup_{i=1}^{n} \begin{matrix} \lambda_i G_i(U_i) \\ -G_i(U_0) \end{matrix}$$

$$\text{if} \bigcup_{i=1}^{n} U_i \in B_\delta(U_0)$$

Since $\bigcup_{i=1}^{n} \left[F_i(G_i(U_i))\right] = \bigcup_{i=1}^{n} U_i$, this can be written as

$$\bigcup_{i=1}^{n} G_i(U_i) - G_i(U_0) \le \bigcup_{i=1}^{n} \frac{1}{\lambda} U_i - U_0$$

If

$$\bigcup_{i=1}^{n} U_i \in B_\delta(U_0) \tag{3.6}$$

which means that $\bigcup_{i=1}^{n} G_i$ is continuous at U_0. Since U_0 is an arbitrary point in $\bigcup_{i=1}^{n} F_i(S)$, it follows that $\bigcup_{i=1}^{n} G_i$ is continuous on $\bigcup_{i=1}^{n} F(S)$. We will now show that $\bigcup_{i=1}^{n} G_i$ is different at U_0.

Since

$$\bigcup_{i=1}^{n} \left[G_i(F_i(Z_i))\right] = \bigcup_{i=1}^{n} Z_i, Z_i \in S$$

The chain rule implies that if $\bigcup_{i=1}^{n} G_i$ is differentiable at U_0, then

$$\bigcup_{i=1}^{n} G_i'(U_0) F_i'(Z_0) = I$$

Therefore, if $\bigcup_{i=1}^{n} G_i$ is differentiable at U_0, the differentiable matrix of $\bigcup_{i=1}^{n} G_i$ must be

$$\bigcup_{i=1}^{n} G_i'(U_0) = \bigcup_{i=1}^{n} [F_i(X_0)]^{-1}$$

So to show that $\bigcup_{i=1}^{n} G_i$ is differentiable at U_0, we must show that if

$$\bigcup_{i=1}^{n} H_i(U_i) = \frac{\bigcup_{i=1}^{n} G_i(U_i) - \bigcup_{i=1}^{n} G_i(U_o) - \bigcup_{i=1}^{n}\left[F_i(Z_0)\right]^{-1} \bigcup_{i=1}^{n}(U_i - U_0)}{\bigcup_{i=1}^{n}[U_i - U_0]}$$

For

$$\bigcup_{i=1}^{n} U_i \neq U_0 \tag{3.7}$$

Then,

$$\lim_{U_i \to U_0} \bigcup_{i=1}^{n} H_i(U_i) = 0 \tag{3.8}$$

Since $\bigcup_{i=1}^{n} F_i$ is one to one on S and $\bigcup_{i=1}^{n} F_i'(G_i(U_i)) = \bigcup_{i=1}^{n} U_i$, it follows that $\bigcup_{i=1}^{n} U_i \neq U_0$, then $\bigcup_{i=1}^{n} G_i(U_i) \neq \bigcup_{i=1}^{n} G_i(U_0)$. Therefore, we can multiply the numerator and denominator of Equation (3.7) by $\bigcup_{i=1}^{n} G_i(U_i) - G_i(U_0)$ to obtain

$$\bigcup_{i=1}^{n} H_i(U_i) = \frac{\bigcup_{i=1}^{n} \left|G_i(U_i) - G_i(U_o)\right|}{\bigcup_{i=1}^{n} |U_i - U_0|}$$

$$\left(\frac{\bigcup_{i=1}^{n} G_i(U_i) - \bigcup_{i=1}^{n} G_i(U_o) - \bigcup_{i=1}^{n} \left[F_i(Z_0)\right]^{-1} \bigcup_{i=1}^{n}(U_i - U_0)}{\bigcup_{i=1}^{n} \left|G_i(U_i) - G_i(U_o)\right|} \right)$$

$$= \frac{\bigcup_{i=1}^{n} \left|G_i(U_i) - G_i(U_o)\right|}{\bigcup_{i=1}^{n} |U_i - U_0|} \left[F_i(Z_0)\right]^{-1}$$

$$\left(\frac{\bigcup_{i=1}^{n}(U_i - U_0) - F_i'(Z_0)\left(G_i(U_i)\right) - \bigcup_{i=1}^{n} G_i(U_0)}{\bigcup_{i=1}^{n} \left|G_i(U_i) - G_i(U_o)\right|} \right)$$

If $0 < \bigcup_{i=1}^{n} |U_i - U_0| < \delta$. Because of Equation (3.6), this implies

$$\bigcup_{i=1}^{n} \left|H_i(U_i)\right| \leq \bigcup_{i=1}^{n} \frac{1}{\lambda_1} \left\| \left[F_i(Z_0)\right]^{-1} \right\| \left\| \frac{\bigcup_{i=1}^{n}(U_i - U_0) - F_i'(Z_0)\left(G_i(U_i)\right) - \bigcup_{i=1}^{n} G_i(U_0)}{\bigcup_{i=1}^{n} \left|G_i(U_i) - G_i(U_0)\right|} \right\|$$

If

$$\bigcup_{i=1}^{n} |U_i - U_0| < \delta$$

Now let

$$\bigcup_{i=1}^{n} H_{i,j}(U_i) = \frac{\bigcup_{i=1}^{n}(U_i - U_0) - \bigcup_{i=1}^{n} F_i'(X_0)\left(G_i(U_i) - G_i(U_0)\right)}{\bigcup_{i=1}^{n} \left|G_i(U_i) - G_i(U_0)\right|}$$

To complete the proof of Equation (3.8), we must show that $\lim_{U_i \to U_0} H_{i,j}(U_i) = 0$. Since $\bigcup_{i=1}^{n} F_i$ is differentiable at Z_0 we know that if

$$\bigcup_{i=1}^{n} H_{i,k}(Z_i) = \lim_{Z_i \to Z_0} \bigcup_{i=1}^{n} H_{i,j}(U_i)$$

$$= \frac{\bigcup_{i=1}^{n}\left(F_i(Z_i)\right) - \bigcup_{i=1}^{n} F_i(Z_0) - \bigcup_{i=1}^{n} F_i'(Z_0)(Z - Z_0)}{\bigcup_{i=1}^{n} |Z - Z_0|} \tag{3.9}$$

Then,

$$\lim_{Z_i \to Z_0} H_{i,k}\left(Z_i\right) = 0$$

Since $\bigcup_{i=1}^{n} F_i(G_i\left(U_i\right) = \bigcup_{i=1}^{n} U_i$ and

$$Z_0 = \bigcup_{i=1}^{n} G_i\left(U_0\right)$$

$$\bigcup_{i=1}^{n}\left(H_{i,j}\left(U_i\right)\right) = \bigcup_{i=1}^{n}\left(H_{i,k}\left(G_i\left(U_i\right)\right)\right)$$

Now, suppose for $\varepsilon > 0, \exists \delta_j > 0 \bigcup_{i=1}^{n}\left|H_{i,k}\left(Z_i\right)\right| < \varepsilon,$ if

$$0 < \bigcup_{i=1}^{n}\left|Z_i - X_0\right| = \bigcup_{i=1}^{n}\left|Z_i - G_i\left(U_0\right)\right| < \delta_j$$

Since $\bigcup_{i=1}^{n} G_i$ is continuous at U_0, there is a $\delta_{i,k} \in (0, \delta)$ such that

$$\bigcup_{i=1}^{n}\left|G_i\left(U_i\right) - G_i\left(U_0\right)\right| < \delta_j$$

if

$$0 < \bigcup_{i=1}^{n}\left|U_i - U_0\right| < \delta_{i,k}$$

This and Equation (3.11) imply that

$$\bigcup_{i=1}^{n}\left|H_{i,k}\left(U_i\right)\right| = \bigcup_{i=1}^{n}\left|H_{i,k} G_i\left(U_i\right)\right| < \varepsilon$$

If $0 < \bigcup_{i=1}^{n}\left|U_i - U_0\right| < \delta_{i,k}$

Since this implies (3.9), $\bigcup_{i=1}^{n} G_i$ is differentiable at X_0.

Since U_0 is an arbitrary member of $\bigcup_{i=1}^{n} F_i\left(N_i\right),$ we can now drop the zero subscript and conclude that $\bigcup_{i=1}^{n} G_i$ is continuous and differentiable on $\bigcup_{i=1}^{n} F_i\left(N_i\right),$ and

$$\bigcup_{i=1}^{n}[G_i'\left(U_i\right)] = \bigcup_{i=1}^{n}\left[F_i'\left(Z_i\right)\right]^{-1}, \bigcup_{i=1}^{n} U_i \in \bigcup_{i=1}^{n} F_i\left(N_i\right)$$

Hence,

$$G'\left(N\right) = \bigcup_{i=1}^{n} G_i\left(N_i\right) = \left[\bigcup_{i=1}^{n}\left\{F_i\left(Z_i\right)\right\}\right]^{-1}$$

Where

$$\bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} F_i\left(Z_i\right), \bigcup_{i=1}^{n} U_i \in \bigcup_{i=1}^{n} F_i\left(N_i\right)$$

and hence the proof

Corollary 3.3: If $\bigcup_{i=1}^{n} F_i$ is continuously differentiable on a neighborhood of Z_0 and $\bigcup_{i=1}^{n} J_i F_i\left(Z_0\right) \neq 0,$ then, there is an open neighborhood $\bigcup_{i=1}^{n} N_i$ of Z_0 on which the conclusion of the main result holds.

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