ISSN 2581-3463
AJMS
Asian Journal of Mathematical Sciences

RESEARCH ARTICLE

On Generalized Classical Fréchet Derivatives in the Real Banach Space

Chigozie Emmanuel Eziokwu

Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria

Received: 01-08-2020; Revised: 20-09-2020; Accepted: 10-10-2020

ABSTRACT

In this work, we reviewed the Fréchet derivatives beginning with the basic definitions and touching most of the important basic results. These results include among others the chain rule, mean value theorem, and Taylor's formula for differentiation. Obviously, having clarified that the Fréchet differential operators exist in the real Banach domain and that the operators are clearly continuous, we then in the last section for main results developed generalized results for the Fréchet derivatives of the chain rule, mean value theorem, and Taylor's formula among others which become highly useful in the analysis of generalized Banach space problems and their solutions in R^n.

Key words: Banach space, continuity, Fréchet derivatives, mean value theorem, Taylor's formula
2010 Mathematics Subject Classification: 46BXX, 46B25

THE USUAL FRECHET DERIVATIVES

Given x a fixed point in a Banach space X and Y another Banach space, a continuous linear operator S: $X \to Y$ is called the Frechet derivative of the operator T: $X \to Y$ at x if

$$T(x+\Delta x)-T(x)=S(\Delta x)-\varphi(x,\Delta x)$$

and

$$\lim_{\|x\|\to 0}\frac{\|\varphi(x,\Delta x)\|}{\|\Delta x\|}=0$$

Or equivalently,

$$\lim_{\|\Delta x\|\to 0}\frac{\|T(x+\Delta x)-T(x)-S(\Delta x)\|}{\|\Delta x\|}=0$$

This derivative is usually denoted by $dT(x)$ or $T'(x)$ and T is Frechet differentiable on its domain if $T'(x)$ exists at every point of the domain as in Abdul[1] and Argyros[2].

Remark: If $X = R$, $Y = R$, then the classical derivative $f'(x)$ of real function f: $R \to R$ at x

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Is a number representing the slope of the graph of the function f at x where the Frechet derivative of f is not a number but a linear operator on R into R. Existence of $f'(x)$ implies the existence of the Frechet derivative[3] as the two are related by

$$f(x+\Delta x)-f(x)=f(x)\Delta x+(\Delta x)g(\Delta x)$$

Address for correspondence:
Chigozie Emmanuel Eziokwu
E-mail: okereemm@yahoo.com

While S is the operator which multiplies every δx by the number $f'(x)$. In elementary calculus the derivative at x is a local approximation of f in the neighborhood of x while the Frechet derivative is interpreted as the best local linear approximation. It is clear from definition that if T is linear, then the Frechet derivative is linear as well, that is,

$$dT(x) = T'(x)$$

THEOREM 1.1:[4]
If an operator has the Frechet derivative at a point, then it has the Gateaux derivative at that point and both derivatives have equal values.

THEOREM 1.2:[5]
Let Ω be an open subset of X and $T:\Omega \to Y$ have Frechet derivative at an arbitrary point a of Ω. Then T is continuous at a. This means that every Frechet differentiable operator defined on an open subset of a Banach space is continuous.

THEOREM 1.3(CHAIN RULE):[1,6]
Let A, B, and C be real Banach spaces. If S: $A \to B$ and T: $B \to C$ are Frechet differentiable at x and $U'(x) = T'(S(x))S'(x)$. Then, the higher order Frechet derivatives for real $U = T^{\circ}S$ can successively be generated iteratively such that

$$U^{(n)}(x) = T^{(n)}(S(x))S^{n}(x)$$

For $n \geq 2$ and integer.

THEOREM 1.5 (IMPLICIT FUNCTION THEOREM)[1,7,8]
Suppose that X, Y, and Z are Banach spaces, C an open subset of $X \times Y$ and T: $C \to Z$ is continuous, suppose further that for some $(x_1, y_1) \in C$

i. $T(x_1, y_1) = 0$
ii. The Frechet derivative of $T(.,.)$ when x is fixed is denoted by $T_y(x, y)$ called the partial Frechet derivative with respect to y, exists at each point in a neighborhood of (x_1, y_1) and is continuous at (x, y).
iii. $\left[T_y(x_1, y_1)\right]^{-1} \in B(z, y)$ then there is an open subset of X containing x and a unique continuous mapping y: $D \to Y$ such that $T(x, y(x)) = 0$ and $y(x_1) = y_1$

Corollary 1.6: If in addition to theorem 1.5 $T_x(x, y)$ also exists in the open set, and is continuous at (x_1, y_1). Then, F: $x \to y(x)$ has Frechet derivative at x_1 given by

$$F'(x) = -\left[T_y(x_1, y_1)\right]^{-1} T_x(x_1, y_1)$$

THEOREM 1.7 (Taylor's Formula for differentiation)[1,9,10]
Let $T:\Omega \subset X \to Y$ and let $[a, a+\Delta x]$ be any closed segment lying in Ω. If T is Frechet differentiable at a, then

$$T(a+\Delta x) = T'(a)\Delta x + (\Delta x)\varepsilon(\Delta x)$$

$$\lim_{\Delta x \to 0} \varepsilon(\Delta x) = 0$$

and

$$T(a+h) = T(a) + T'(a)\Delta x + \frac{1}{2}\left(T''(a)\Delta x\right) + \|\Delta x\|^2 \varepsilon(\Delta x)$$

$$\lim_{\Delta x \to 0} \varepsilon(\Delta x) = 0$$

For twice differentiable functions.

MAIN RESULTS ON GENERALIZED FRECHET DERIVATIVES

Let x be a fixed point in the real Banach space. Also let the continuous linear operator S: $X \rightarrow Y$ be a real Frechet derivative of the operator T: $X \rightarrow Y$ such that

$$\lim_{\|\Delta x\| \to 0} \frac{\|T(x+\Delta x) - T(x) - S(\Delta x)\|}{\|\Delta x\|} = 0$$

Then, the higher order Frechet derivative successively can be generated in an iterative manner such that

$$\lim_{\sum_{i=1}^{n}\|\Delta x_i\| \to 0} \frac{\left\| T\left[x + \sum_{i=1}^{n}(\Delta x_i)\right] - T\left[x + \sum_{i=1}^{n-1}(\Delta x_i)\right] - S\left(\sum_{i=1}^{n}(\Delta x_i)\right) \right\|}{\left\| \sum_{i=1}^{n} \Delta x_i \right\|}$$

$n \geq 2$ and an integer.

THEOREM 2.1 (CHAIN RULE): Let A, B, and C be a unitary spaces, if S: $A \rightarrow B$ and T: $B \rightarrow C$ are Frechet differentiable at z and $u'(x) = u(s(x)) \circ s'(x)$. Then, the higher order Frechet derivative for $U^{(n)}(x)$ can be generated with $U = S \circ T$ generating $\varphi^{(n)}(z) = U^{(n)}(z)$ if and only if

$$\lim_{\left\|\sum_{i=1}^{n}\Delta z_i\right\| \to 0} \frac{\left\| U\left[z + \left(\sum_{i=1}^{n}\Delta z_i\right) - U\left[z - \left(\sum_{i=1}^{n-1}\Delta z_i\right) - \varphi\left(\sum_{i=1}^{n}\Delta z_i\right)\right]\right] \right\|}{\left\| \sum_{i=1}^{n} \Delta x_i \right\|}$$

$$= \lim_{\left\|\sum_{i=1}^{n}\Delta x_i\right\| \to 0} \frac{\left\| S \circ T\left[x + \left(\sum_{i=1}^{n}\Delta x_i\right)\right] - S \circ T\left[x - \left(\sum_{i=1}^{n-1}\Delta x_i\right) - \varphi\left(\sum_{i=1}^{n}\Delta x_i\right)\right] \right\|}{\left\| \sum_{i=1}^{n} \Delta x_i \right\|}$$

THEOREM 2.2 [Generalized Frechet Mean Value theorem]: Let T: $A \rightarrow B$ where A is an open convex set containing a, b, and c is a normed space. $T^{(n)(x)}$ exists for each $a \in [a,b]$ and $T^{(n-1)}(x)$ is continuous on [a, b], then

$$\left\|T^{(n-1)}(b) - T^{(n-1)}(a)\right\| \leq_{x\in[a,b]}^{\sup} \left\|T^{(n)}(a)\right\| \left\|T^{(n-2)}(b) - T^{(n-2)}(a)\right\|$$

THEOREM 2.3 [Generalized Implicit function theorem]

Suppose that A, B, and C are real Banach spaces, D is an open subset of $A \times B$ and T: $D \rightarrow C$ is continuous. Suppose further that for some $(a,b) \in D$, then

i. $T^{(n)}(a,b) = 0$

ii. The nth Frechet derivative of T (.,.) where x is fixed and denoted by $T_{b_1}^{(n)}(a_1, b_1)$ called the n[th] partial derivative with respect to b exists at each point in a neighborhood of (a_1, b_1) and is continuous at a_1, b_1

iii. $\left[T_x^{(n)}(a_1, b_1)\right]^{-1} \in B(C,B)$ then there is a subset E of A containing a_1 and a unique continuous mapping S: $E \rightarrow C$ such that $T^{(n)}(a_1, b_1(a_1)) = 0$ and $S^{(n)}(a_1) = b_1$

Corollary 2.4: If the addition to conditions of theorem 2.3, $T_a^{(n-1)}(a,b)$ also exists on the open set, and is continuous at (a_1, b_1), then F: $a\to b$ (a) has the nth Frechet derivative at a_1 given by

$$F^{(n)}(a_1) = -\left[T_n^{(n-1)}(a,b)\right]^{-1} T_a^{(n-1)}(a_1,b_1)$$

THEOREM 2.5 [Taylors formula for nth Frechet differentiable functions]
Let $T:\Omega\subset X\to Y$ and $[a,a+n\Delta x]$ be any closed segment lying in Ω. If T is differentiable in Ω and nth differentiable at a, then

$$T(a+n\Delta x) = T(a) + T'(a)\Delta x + \frac{1}{2}\left(T^n(a)\Delta x\right)\Delta x + \cdots + \frac{1}{n!}\left(T^{(n)}(\Delta x)^{(n)}\right) + \|\Delta x\|^n \varepsilon(\Delta x)$$

where

$$\lim_{\Delta x\to\infty} \varepsilon(\Delta x) = 0$$

PROOF OF MAIN RESULTS

Proof of Theorem 2.1 (chain rule)

Let $x,\Delta x\in X$ and suppose U^n (x) can be generated with $U = S\circ T$ such that the generalized Frechet derivative

$$\varphi^n(x) = U^n(x) = U\left[x + \left(\sum_{i=1}^{n}\Delta x_i\right)\right] - U\left(\sum_{i=1}^{n}\Delta x_i\right)$$

$$= T\left[S\left(x_0 + \sum_{i=1}^{n}\Delta x_i\right)\right] - T\left[S\left(\sum_{i=1}^{n}\Delta x_i\right)\right] = T\left[x + \sum_{i=1}^{n}\Delta y_i\right] - T\left(\sum_{i=1}^{n}\Delta y_i\right)$$

where

$$\sum_{i=1}^{n}\Delta x_i = S\left(x + \sum_{i=1}^{n}\Delta x_i\right) - S\left(\sum_{i=1}^{n}\Delta x_i\right)$$

Thus

$$\left\| U(x+\Delta x_i) - U\left(\sum_{i=1}^{n}\Delta x_i\right) - T^n(x_0)x \right\| = \sigma\|(z)\|$$

since

$$\left\|\sum_{i=1}^{n}\Delta x_i - S^n(x)\sum_{i=1}^{n}\Delta x_i\right\| = \sigma'(\|x\|)$$

We get

$$\left\| U\left(x + \sum_{i=1}^{n}\Delta x_i\right) - U\left(\sum_{i=1}^{n}\Delta x_i\right) - T^n(x_0)S^n(x)\Delta x \right\|$$

$$= \left\|\left[U\left(x + \sum_{i=1}^{n}\Delta x_i\right) - U\left(\sum_{i=1}^{n}\Delta x_i\right)\right] - T^n(y)x - T^n(y)S^n(x)\Delta x\right\|$$

In view of the fact that S is continuous at x, we obtain

$$\left\|\sum_{i=1}^{n}\Delta x_i\right\| = \sigma(\|\Delta x\|)$$

therefore

$$\varphi^n(x) = \varphi^n(x) = \lim_{\sum_{i=1}^{n}\|\Delta x_i\|\to 0} \frac{\left\| S\circ T\left[x+\left(\sum_{i=1}^{n}\Delta x_i\right) - S\circ T\left[x-\left(\sum_{i=1}^{n-1}\Delta x_i\right)-\varphi\left(\sum_{i=1}^{n}\Delta x_i\right)\right]\right]\right\|}{\sum_{i=1}^{n}\|\Delta x_i\|}$$

conversely

$$\lim_{\sum_{i=1}^{n}\|\Delta x_i\|\to 0} \frac{\left\| S\circ T\left[x+\left(\sum_{i=1}^{n}\Delta x_i\right) - S\circ T\left[x-\left(\sum_{i=1}^{n-1}\Delta x_i\right)-\varphi\left(\sum_{i=1}^{n}\Delta x_i\right)\right]\right]\right\|}{\sum_{i=1}^{n}\|\Delta x_i\|}$$

implies

$$T\circ S\left[x+\sum_{i=1}^{n}\Delta x_i\right] - T\circ S(x)$$

$$= T\circ S\left[\sum_{i=1}^{n}\Delta x_i\right] + \varphi\left[x_i\sum_{i=1}^{n}\Delta x_i\right] = T\circ S\left[\sum_{i=1}^{n}\Delta x_i\right] + \varphi\left[\sum_{i=1}^{n}\Delta x_i\right]$$

and

$$T\circ S\left[x+\sum_{i=1}^{n}\Delta x_i\right] + S\circ T(x)$$

$$= T\circ S\left[\sum_{i=1}^{n-1}\Delta x_i\right] + \varphi\left[x-\sum_{i=1}^{n}\Delta x_i\right] = T\circ S\left[\sum_{i=1}^{n}\Delta x_i\right] + \varphi\left[\sum_{i=1}^{n}\Delta x_i\right]$$

$$= \lim_{\sum\|\Delta x_i\|\to 0} \frac{\left\| T\circ S\left[x+\sum\Delta x_i\right] - T\circ S\left[x-\sum_{i=1}^{n-1}\Delta x_i\right] - \varphi\left[\sum_{i=1}^{n}\Delta x_i\right]\right\|}{\sum_{i=1}^{n-1}\|\Delta x_i\|}$$

$$= \lim_{\sum\|\Delta x_i\|\to 0} \frac{\left\|\sum_{i=1}^{n-1}\Delta x_i\right\|}{\sum\|\Delta x_i\|} = \lim_{\Delta x_n\to 0}\frac{0}{\Delta x_i} = 0 \quad \text{(By L'Hospital Rule)}$$

Hence, $U^n(x) = \varphi^n(x)$ is Frechet differentiable and the proof is complete.

PROOF OF THE GENERALIZED FRECHET MEAN VALUE THEOREM

Let T: $K\to B$ where K an open convex set containing is a and b. B is a normal space and $T^{(n)}(x)$ exists for each x in $[a, b]$ and $T'(x)$ is continuous in $[a, b]$ such that

$$\|T(b) - T(a)\| \leq \sup_{x\in[a,b]} \|T'(x)\| \|b-a\|$$

Then by induction for the nth complex iterative Frechet derivative of T, the mean value theorem becomes

$$\left\|T^{(n-1)}(b) - T^{(n-1)}(a)\right\| \leq \sup_{x\in[a,b]} \left\|T^{(n)}(x)\right\| \left\|T^{(n-2)}(b) - T^{(n-2)}(a)\right\|$$

Proof of Theorem 2.3 [generalized implicit function theorem]

For the sake of convenience, we may take $x_1 = 0$ and $x_2^* = 0$. let

$$A = \left[T_{x_2}^{(n-1)}(0,0) \in B(C,B)\right]$$

Since D is an open set containing $(0,0)$, we find that

$$D \in D_{x_1} = \{x_2 \in C/(x_1, x_2) \in D\}$$

For all sufficiently small say $\|x_1\| \leq \delta$. For each x_1 having this property, we define a function $S^{(n-1)}(x_1, x_2)$ $D_{x_1} \to C$ by $S^{(n-1)}(x_1, x_2) = x_2 - FT^{(n-2)}(x_1, x_2)$. To prove the theorem, we must prove the existence of a fixed point for $S^{(n-1)}(x_1, x_2)$ under the condition that $\|x_1\|$ is sufficiently small. Continuity of the mapping $x_1 \to x_2(x_1)$ and $x_2(x_1) \to x_2^*$. Now,

$$S_{x_2}^{(n-1)}(x_1, x_2)(U) = U - T_{x_2}(x_1, x_2)(U)$$

and

$$FF^{-1} = FT_{x_2}^{(n-1)}(0,0)$$

Therefore, assumptions on $T^{(n-1)}$ guarantees the existence of $S^{(n-1)}(x_1, x_2)$ for sufficiently small $\|x_1\|$ and $\|x_2\|$ and

$$S^{(n-1)}(x_1, x_2)(U) = F\left[T_{x_2}^{(n-1)}(0,0) - T_{x_2}^{(n-1)}(x_1, x_2)(U)\right]$$

hence

$$\left\|S^{(n-1)}(x_1, x_2)\right\| \leq \|F\| \left\|T_{x_2}^{(n-1)}(0,0) - T_{x_2}^{(n-1)}(x_1, x_2)\right\|$$

Since, $T_{x_2}^{(n-1)}$ is continuous at (0, 0) there exists a constant $L < 0$ such that

$$\left\|S^{(n-1)}(x_1, x_2)\right\| \leq L \tag{3.3.1}$$

Or sufficiently small $\|x_1\|$ and $\|x_2\|$, we say that $\|x_1\| \leq \varepsilon_1 \leq \delta$ and $\|x_2\| \leq \varepsilon_2$. Since $T^{(n-1)}$ is continuous at (0, 0), there exists an $\varepsilon \leq \varepsilon_1$ such that

$$\left\|S^{(n-1)}(x_{1,0})\right\| \left\|FT^{(n)}(x_{1,0})\right\| \leq \varepsilon_2(1-L) \tag{3.3.2}$$

For all x_1 with $\|x_1\| \leq \varepsilon$. We now show that $S^{(n-1)}(x_1,.)$ maps the closed ball $\bar{S}_{\varepsilon}^{(n-1)}(0) = \{x_2 \in B/\|x_2\| \leq \varepsilon_2\}$ into itself. For this let $\|x_1\| \leq \varepsilon$ and $\|x_2\| \leq \varepsilon_2$. Then by the Mean Value theorem and (3.3.1), (3.3.2), we have

$$\left\|S^{(n-1)}(x_1, x_2)\right\| \leq \left\|S^{(n-1)}(x_1, x_2) - S^{(n-1)}(x_1, 0)\right\| + \left\|S^{(n-1)}(x, 0)\right\|$$

$$\leq \sup_{0 \leq \alpha \leq 1} \left\|S_{x_2}^{(n)}(x_1, x_2^*)\right\| \|x_2\| + \left\|S^{(n-1)}(x_1, 0)\right\| \leq L_{\varepsilon_2} + \varepsilon_2(1-L) = \varepsilon_2$$

Therefore, for $\|x_1\| \leq \varepsilon, S^{(n-1)}(x_1,.) : \bar{S}_{\varepsilon_2}^{(n-1)}(0)$. Also for $x_2^*, x_2^{**} \in \bar{S}_{\varepsilon_2}(0)$; we obtain by the mean value theorem of section 2.2 and equation (3.3.1)

$$\left\|S^{(n-1)}(x_1, x_2^*) S^{(n-1)}(x_1, x_2^{**})\right\| \leq \sup_{\|x_2\| \leq \varepsilon_2} \left\|S_{x_2}^{(n-1)}(x_1, x_2)\right\| \left\|x_2^* - x_2^{**}\right\| \leq L \left\|x_2^* - x_2^{**}\right\|$$

The Banach contraction mapping theorem guarantees that for each x_1 with $\|x_1\| \leq \varepsilon$ there exists a unique $x_2(x_1) \in S_{\varepsilon_2}^{(n-1)}(0)$ such that

$$x_2(x_1) = S^{(n-1)}(x_1, x_2(x_1)) = x_2(x_1) - FT^{(n-1)}(x_1, x_2(x_1))$$

That is,

$$T^{(n-1)}(x_1, x_2(x_1)) = 0$$

By the uniqueness of x_2, we have that x_2 (0) = 0 since

$T^{(n-1)}(0,0)=0$

Finally, we show that $(x_1 \to x_2(x_1))$ is continuous for if $\|x_1^*\|<\varepsilon$ and $\|x_1^{**}\|<\varepsilon$, and then selecting $x_2^0 = x_1, x_2(x_1^{**})$ and $x_2^* = S^{(n-1)}(x_1^*, x_2^0)$. We have by the error bound for fixed point iteration on the mapping $S^{(n-1)}(x_1,.)$

$$\left\|x_2(x_1^{**}) - x_2(x_1)\right\| \le \frac{1}{1-L}\left\|x_2^0 - x_2^*\right\|$$

We can write

$$x_2^0 - x_2^* = x_2(x_1^{**}) - S^{(n-1)}(x_1^*, x_2(x_1^{**}))$$
$$= S^{(n-1)}(x_1^*, x_2(x_1^{**})) - T^{(n-1)}(x_1^*, x_2(x_1^{**}))$$
$$= F\left[T^{(n-1)}(x_1^*, x_2(x_1^{**})) - T^{(n-1)}(x_1, x_2(x_1^{**}))\right]$$

Therefore, by continuity of $T^{(n-1)}$, $\left\|x_2\left(x_1^{**} - x_2(x_1^{**})\right)\right\|$ can be made arbitrary small for $\|x_1^{**} - x_1^*\|$ sufficiently small and hence the proof.

Proof of Corollary 2.4

We set $x_1 = x_1^* + \Delta x_1$ and $G^{(n)}\Delta x_1^* = F^{(n)}(x_1) - x_1$. Then $G^{(n)}(0) = 0$ and

$$\left\|G^{(n-1)}(\Delta x_1^*) + \left[T_{x_2}^{(n-1)}(x_1 - x_2)\right]^{-1}\Delta x_1^*\right\|$$
$$\le \left\|\left[T_{x_2}^{(n-1)}(x_1^*, x_2^*)\right]^{-1}\right\|\left\|T_{x_2}^{(n-1)}(x_1,x_2)\Delta x_1^* G^{(n-1)}(\Delta x_1^*) + T_{x_1}^{(n-1)}(x_1,x_2)\Delta x_1^*\right\|$$

and

$$T_{x_2}^{(n-1)}(x_1,x_2)\Delta x_1^* G^{(n-1)}(\Delta x_1^*) + T_{x_1}^{(n-1)}(x_1,x_2)\Delta x_1^*$$
$$= -T^{(n-1)}\left(x_1^* + \Delta x_1^*, x_2 + G^{(n-1)}(\Delta x_1^*)\right)\Delta x_1^* + T^{(n-1)}(x_1^*, x_2^*)$$
$$+T_{x_2}^{(n-1)}G^{(n-1)}(\Delta x_1^*) +_{x_1}^{(n-1)}(x_1,x_2)\Delta x_1^*$$

If O_1, O_2 are numbers in (0,1), then

$$T_{x_2}^{(n-1)}(x_1,x_2)\Delta x_1^* G^{(n-1)}(\Delta x_1^*) + T_{x_1}^{(n-1)}(x_1,x_2)\Delta x_1^*$$
$$\le \sup\left\|T_{x_1}^{(n-1)}\left(x_1^* + O_1\Delta x_1^*, x_2 + O_2 G^{(n-1)}(\Delta x_1^*) - T_{x_1}^{(n-1)}(x_1^*, x_2^*)\right)\right\|\left\|\Delta x_1^*\right\|$$
$$+\sup_{O_1 O_2}\left\|T_{x_1}^{(n-1)}\left(x_1^* + O_1\Delta x_1^*, x_2 + O_2 G^{(n-1)}(\Delta x_1^*) - T_{x_2}^{(n-1)}(x_1^*, x_2^*)\right)\right\|\left\|G^{(n-2)}(\Delta x_1^*)\right\|$$

Thus applying continuity od $T_{x_1}^{(n-1)}$, $T_{x_2}^{(n-1)}$ for $\varepsilon > 0$, we find that $\delta = \delta(\varepsilon)$ such that on $\|x_1 - x_1^*\| < \delta$, we have

$$\left\| G^{(n-1)}\left(\Delta x_1^*\right) + \left[T_{x_2}^{(n-1)}\left(x_1^*, x_2^*\right)\right]^{-1} T_{x_1}^{(n-1)}\left(x_1, x_2\right)\Delta x_1^* \right\|$$

$$\leq \frac{\left[T_{x_2}^{(n-1)}\left(x_1^*, x_2^*\right)\right]^{-1} \varepsilon \left\| 1 + \left[T_{x_2}^{(n-1)}\left(x_1^*, x_2^*\right)\right]^{-1} \right\| \left[T_{x_1}^{(n-1)}\left(x_1^*, x_2^*\right)\right] \left\| \Delta x_1^* \right\|}{1 - \left\| \left[T_{x_2}^{(n-1)}\left(x_1, x_2\right)\right]^{-1} \right\|}$$

The coefficient of $\left\|\Delta x_1^*\right\|$ can be as small as required as $\left\|\Delta x_1^*\right\| \to 0$. Thus,

$$\left\| F^{(n)}\left(x_1\right) - F^{(n)}\Delta x_1^{**} - \left[T_{x_2}^{(n-1)}\left(x_1, x_2\right)\right]^{-1} T_{x_1}^{(n-1)}\left(x_1, x_1^*\right)\left(x_1 - x_2^*\right) \right\| = \sigma\left(\left\|x_1 - x_1^*\right\|\right)$$

Hence,

$$F^{(n)}\left(x_1^*\right) = -\left[T^{(n-1)}\left(x_1, x_2\right)\right]^{-1} T_{x_2}^{(n-1)}\left(x_1^*, x_2^*\right)$$

Proof of Taylor's formula for nth Frechet differentiable function

The proof of this theorem can be generated as in Carton[11] and Nasheed.[12]

REFERENCES

1. Abdul HS. Applied Functional Analysis. New York, USA: Rutgery University; 2000. p. 201-17.
2. Argyros IK. Approximate Solution of Operator Equations with Applications. United States: Cameron University USA; 2005. p. 6-12.
3. Athanasios GK. Advanced Differential Equations. Florida: Malner Publishing Company, Inc.; 1971.
4. Griffel DH. Applied Functional Analysis. Chichester: Ellis Harwood Limited Publishers; 1981.
5. Fréchet M. On some points of functional calculation. Makes Circ Matt Palermo 1905; 22: 1-74.
6. Tapia RA. The differentiation and integration of non-linear operators. In: Rall LB, editor. Non Linear Functional Analysis and Applications. New York, London: Academic Press; 1971. p. 45-108.
7. Milne RD. Applied Functional Analysis. Bostonk, London, Melbourne: Pitman Academic Publishing Program; 1980.
8. Cartain RF, Pritchard AJ. Functional Analysis in Modern and Applied Mathematics. London, New York: Academic Press; 1977.
9. William FT. Introduction to Real Analysis. Nigeria: National Mathematical Centre Abuja; 2010.
10. William FT. Instruction Solution Manual on Introduction to Real Analysis. Abuja, Nigeria: National Mathematical Centre; 2010.
11. Carton H. Differential Calculus. Paris; Herman/Kershaw; 1971.
12. Nashed MZ. Differentiability and related properties of non-linear operators: Some aspects of the role of differentials in non-linear functional analysis In: Rall LB, editor. Non Linear Functional Analysis and Applications. New York, London: Academic Press; 1971. p. 103-309.