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On The Generalized Topological Set Extension Results Using The Cluster Point Approach

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ABSTRACT

In this work, we seek generalized finite extensions for a set of real numbers in the topological space through the cluster point approach. Basically, we know that in the topological space, a point is said to be a cluster point of a subset X if and only if every open set containing the point say x contains another point of x_1 different from x. This concept with the aid basic known ideas on set theory was carefully used in the definition of linear, radial, and circular types of operators which played the major roles in realizing generalized extension results as in our main results of section three.

Key words: Cluster point, set, topological extensions, topological operators, topological space

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INTRODUCTION [CLUSTER POINT]

Let X be a topological space. A point $x \in X$ is said to be a cluster point (accumulation point, limit point, or derived point) of a subset X_1 of X if and only if every open set G containing x contains a point of X_1 different from x, that is, G open, $x \in G$ implies $\left(G-\{x\}\right)\bigcap X_1 \neq 0$. The set of cluster points of X_1 is called the derived set of X_1 and is denoted by X_1^1.

Definition 1.1:[1-3] Let X be a topological space. A subset X_1 of X is a closed set if and only if it's complement X_1^c is an open set while the closure of X_1 denoted by $\bar{X}$ is the intersection of all closed supersets of X. In other words, if $\{F_i;\ i\in I\}$ is the class of all closed subsets of X containing X_1, then $\bar{X}_1 = \bigcap_i F_i$.

Definition 1.2:[4-9] Let X_1 be a subset of a topological set X. A point $x \in X_1$ is called an interior point of X_1 if x belongs to an open set G contained in X_1 such that $x \in G \subset A$ where G is open. The set of interior points of X_1 denoted by int (X_1) or X_1^0 is called the interior of X_1. The exterior of X_1 written *ext* (X_1) is the interior of the complement of X_1 (i.e.,int X_1^c). The boundary of X_1 written b (X_l) is the set of all points which do not belong to the interior or the exterior of X_1.

Theorem 1.1:[7-9] Let X_1 be a subset of the topological space X. Then, $\bar{X}_1$ the closure of X_1 is the union of X_1 and its set of accumulation points, that is, $\bar{X} = X\bigcup X^1$.

Remark 1.1:[10-12] A subset X_1 of a topological space X is dense in $X_2 \subset X$ if X_2 is contained in the closure of X_1 i.e. $x \in \bar{X}_1$.

Theorem 1.2:[13-15] Let X_1 be any subset of the topological space X. Then, the closure of X_1 is the union of the interior and boundary of X_1, that is, $\bar{X}_1 = X_1^0 \bigcup b\left(X_1\right)$.

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TOPOLOGICAL EXTENSIONS

Let (*X,d*) be a topological space where *X* is any given topological space and *d* is the map or function operating on *X*. Then, the extension map or functional or operator *d,τ* or *T* involved in this work are just abstract maps such as the union, intersection, complementation maps, and all the likes. These make the construction and establishment of main results in this work easier than the existing traditional approaches. However, the raw topological maps used in this work obey the linearity, radial, and circular conditions seen in definition (2.1), (2.2), and (2.3) below.

Linear extensions

An extension map *d,τ,* or *T* which may be a union intersection, complementation is called linear if

i. $f(x_1+x_2)=f(x_1)+f(x_2)$ or more general

$$f\left(\sum_{i=1}^{n}x_i\right)=\sum_{i=1}^{n}f(x_i) \text{ or } \left(\bigcup_{i=1}^{n}A_i\right)^c=\bigcap_{i=1}^{n}\left(A_i^c\right)$$

ii. $(\alpha+\beta)f(x)=\alpha f(x_i)+\beta f(x_i)$ or $\sum_{i=1}^{n}(\alpha_i+\beta_i)f(x_i)=\sum_{i=1}^{n}\alpha_i f(x_i)+\sum_{i=1}^{n}\beta_i f(x_i)$

or $\left(\bigcup_{i=1}^{n}\alpha_i A_i\right)^c=\bigcap_{i=1}^{n}\alpha_i A_i^c$

Theorem 2.1.1: The functional $\sum_{i=1}^{n}f_i$ is an extension of $\sum_{i=1}^{n}f_i$ if $\sum_{i=1}^{n}\bar{f}_i(x_i)=\sum_{i=1}^{n}f_i(x_i)$ for all $x_i \in domf_i$, the space (*X,d*) such that $\bigcup_{i=1}^{n}X_i$ is a dense subset of *X* for any given complete metric space (*Y,d*) then $\sum f_i:\bigcup_{i=1}^{n}X_i\to Y$ is uniformly continuous and has a unique continuous extension $\sum_{i=1}^{n}f_i:X\to Y$ such that $\sum\bar{f}$ is also uniformly

Theorem 2.1.2 [Generalized Tietze's Extension]: Let *X* be a normal space and let X_1 be a closed subspace of *X*. Then

a) Any collection of continuous maps of $\bigcup_{i=1}^{n}X_i$ into the collection of interval $\bigcup_{i=1}^{n}[a_i,b_i]$ of R may be extended to a collection of continuous maps of all of *X* into $\bigcup_{i=1}^{n}[a_i,b_i]$

b) Any continuous collection of maps $\bigcup_{i=1}^{n}X_i$ of *X* into R may be extended to a collection of continuous maps of all of *X* into R.

Note: The above theorem is a generalized nonlinear extension theorem for the space $X=R$, the real's.

The radial topological operator

The radial topological operator *T* is a linear operator that is mapped from domain that is linear into a range that is circular. By this, we mean an operator *T* that satisfies the map

$$T:\alpha\left(x_i+x_j\right)\in X\to\pi x_i^2\in X$$

For any given set *X* such that *α* is any constant and π is the constant 3.147.

Circular topological operator

Given any set *X* and x_i, finite elements of *X*. Then, the map *T* is called a circular operator if *T* satisfies $T:\pi x^2\subset X\to\pi x^2\subset X$ where *π* is the constant 3.147.

MAIN RESULTS

Theorem 3.1: Let $\bigcup_{i=1}^{n} X'$ be the set of cluster points of the set *X*. If

$$\bigcup_{i=1}^{n} X' = \bigcup_{i=1}^{n} x_i', x_i \in \bigcup_{i=1}^{n} X' = \bigcup_{i=1}^{n}\left(\left(B_r(x_i) - \{x_i\}\right) \bigcap_{i=1}^{n} \left(\bigcup_{i=1}^{n} X_i \right) \right) \neq \varnothing$$

Then, the closure $\bigcup_{i=1}^{n} X_i$,

$$\bigcup_{i=1}^{n} \bar{X}_i = \left(\bigcup_{i=1}^{n} X_i \right) \bigcup_{i=1}^{n} \left(\bigcup_{i=1}^{n} X_i' \right)$$

$$= \left(\bigcup_{i=1}^{n} X_i \right) \bigcup_{i=1}^{n} \left(\left(B_r(x_i) - \{x_i\}\right) \bigcap_{i=1}^{n} \left(\bigcup_{i=1}^{n} X_i \right) \right) \neq \varnothing$$

where $\bigcup_{i=1}^{n} X_i \subset X$.

Proof:

Let $\bigcup_{i=1}^{n} x_i \in \left(\left(\bigcup_{i=1}^{n} X_i \right) \bigcup_{i=1}^{n} \left(\bigcup_{i=1}^{n} X_i' \right) \right)^c$ since $\bigcup_{i=1}^{n} x_i \notin \bigcup_{i=1}^{n} X_i', \exists$ an open set $G = \left\{ \bigcup_{i=1}^{n} x_i \right\}$ such that $\bigcup_{i=1}^{n} x_i \in G$ and $G \bigcap \bigcup_{i=1}^{n} X_i = \left\{ \bigcup_{i=1}^{n} x_i \right\} \bigcap \bigcup_{i=1}^{n} X_i = \varnothing$ or $\bigcup_{i=1}^{n} x_i$ however $\bigcup_{i=1}^{n} x_i \notin \bigcup X_i$. Hence in particular $G \bigcap \left(\bigcup_{i=1}^{n} X_i \right) = \varnothing$, we also claim that $G \bigcap \left(\bigcup_{i=1}^{n} X_i \right) = \varnothing$. For if $\left\{ \bigcup_{i=1}^{n} x_i \right\} \in G$ then $\left(\bigcup_{i=1}^{n} x_i \right) \in G$ and $G \bigcap \left(\bigcup_{i=1}^{n} X_i \right) = \varnothing$ where *G* is an open set so $\bigcup_{i=1}^{n} x_i \notin \bigcup_{i=1}^{n} X_i$ and thus $G \bigcap \left(\bigcup_{i=1}^{n} X_i \right) = \varnothing$.

Accordingly,

$$G \bigcap_{i=1}^{n} \left(\bigcup X_i \bigcup \left(\bigcup X_i \right) \right)$$

$$= \left(G \bigcap \left(\bigcup_{i=1}^{n} X_i \right) \bigcup \left(G \bigcup_{i=1}^{n} \left(\bigcup_{i=1}^{n} X_i \right) \right) \right) = \varnothing \cup \varnothing = \varnothing$$

and so

$$G \subset \left(\left(\bigcup_{i=1}^{n} X_i \right) \left(\bigcup_{i=1}^{n} X_i \right) \right)^c$$

Thus, $\bigcup_{i=1}^{n} x_i$ is an interior point set of $\left(\bigcup_{i=1}^{n} X_i \bigcup_{i=1}^{n} \left(\bigcup_{i=1}^{n} X_i \right) \right)^c$ which is, therefore, an open set. Hence, $\left(\bigcup_{i=1}^{n} X_i \right) \bigcup \left(\bigcup_{i=1}^{n} X_i' \right)$ is closed. We now show that

$$\bigcup_{i=1}^{n} \bar{X}_i = \left(\bigcup_{i=1}^{n} X_i \right) \bigcup_{i=1}^{n} \left(\bigcup_{i=1}^{n} X_i' \right)$$

Since $\bigcup_{i=1}^{n} X_i \subset \bigcup_{i=1}^{n} \bar{X}_i$ and $\bar{A}$ is closed, $\bigcup_{i=1}^{n} X_i' \subset \bigcup_{i=1}^{n} \bar{X}_i' \subset \bigcup_{i=1}^{n} \bar{X}_i$ but $\left(\bigcup_{i=1}^{n} X_i\right) \bigcup \left(\bigcup_{i=1}^{n} X_i'\right)$ is a closed set containing $\bigcup_{i=1}^{n} X_i$, so $\bigcup_{i=1}^{n} X_i \subset \bigcup_{i=1}^{n} \bar{X}_i \subset \bigcup X_i'$. Thus

$$\bigcup_{i=1}^{n} \bar{X}_i = \bigcup_{i=1}^{n} X_i \bigcup_{i=1}^{n} \left(\bigcup_{i=1}^{n} X_i\right)$$

$$= \left(\bigcup_{i=1}^{n} X_i\right) \bigcup_{i=1}^{n} \left(B_r(x_i) - \{x_i\} \bigcap_{i=1}^{n} \left(\bigcup_{i=1}^{n} X_i\right)\right) \neq \varnothing$$

and $\bigcup_{i=1}^{n} X_i \subset X$.

Theorem 3.2: Let $\bigcup_{i=1}^{n} X_i$ be any subset of a topological space *X*. Then, the closure of $\bigcup_{i=1}^{n} X$ is the union of the interior and boundary of $\bigcup_{i=1}^{n} X_i$ i.e. $\bigcup_{i=1}^{n} \bar{X}_i = \left(\bigcup_{i=1}^{n} X_i^0\right) \bigcup \left(b \bigcup_{i=1}^{n} X_i\right)$.

Proof: Since

$$X = \operatorname{int} \bigcup_{i=1}^{n} X_i \cup b \bigcup_{i=1}^{n} X_i \bigcup_{i=1} ext \bigcup_{i=1}^{n} X_i$$

Therefore,

$$\left(\operatorname{int}\left(\bigcup_{i=1}^{n} X_i\right) \bigcup b\left(\bigcup_{i=1}^{n} X_i\right)\right)^c = ext\left(\bigcup_{i=1}^{n} X_i\right)$$

and it suffices to show that $\left(\bigcup \bar{X}_i\right)^c = ext\left(\bigcup_{i=1}^{n} X_i\right)$.

Let $\bigcup_{i=1}^{n} X_i \in ext\left(\bigcup_{i=1}^{n} X_i\right)$, then $\exists$ an open G such that $\bigcup_{i=1}^{n} X_i \in G \subset \bigcup_{i=1}^{n} X_i^c$ which implies $G \bigcap_{i=1}^{n} \left(\bigcup_{i=1}^{n} X_i\right) = \varnothing$. So $\bigcup_{i=1}^{n} x_i$ is not a limit point of $\bigcup_{i=1}^{n} X_i$ i.e. $\bigcup_{i=1}^{n} x_i \notin \bigcup_{i=1}^{n} X_i'$ and $\bigcup_{i=1}^{n} x_i \notin \bigcup_{i=1}^{n} X_i$. Hence $\bigcup_{i=1}^{n} x_i \in \left(\bigcup_{i=1}^{n} X_i'\right) \bigcup \left(\bigcup_{i=1}^{n} X_i\right) = \bigcup_{i=1}^{n} \bar{X}_i$. In other words $ext\left(\bigcup_{i=1}^{n} X_i\right) \subset \left(\bigcup_{i=1}^{n} \bar{X}_i\right)^c$. Now assume $\bigcup_{i=1}^{n} x_i \in \left(\bigcup \bar{X}_i\right)^c = \left(\left(\bigcup_{i=1}^{n} X_i\right) \bigcup \left(\bigcup_{i=1}^{n} X_i'\right)\right)^c$. Thus, $\bigcup_{i=1}^{n} x_i \notin \bigcup X_i'$, so $\exists$ an open set *G* such that $\bigcup_{i=1}^{n} x_i \in G$ and $G - \{\bigcup x_i\} \bigcap \left(G - \{\bigcup x_i\}\right) \bigcap \left(\bigcup X_i\right) = \varnothing$. But also $\bigcup_{i=1}^{n} x_i \notin \bigcup_{i=1}^{n} X_i$, So $\left(G \bigcap \{\bigcup x_i\}\right) = \varnothing$ and $\bigcup_{i=1}^{n} x_i \in G \subset \left(\bigcup X_i\right)^c$.

Thus, $\bigcup_{i=1}^{n} x_i \in ext\left(\bigcup_{i=1}^{n} X_i\right)$ and $\left(\bigcup_{i=1}^{n} \bar{X}_i\right) = ext\left(\bigcup_{i=1}^{n} X_i\right)$. Hence, $\bigcup_{i=1}^{n} X_i = \left(\bigcup_{i=1}^{n} X_i^0\right) \bigcup_{i=1}^{n} \left(b\left(\bigcup_{i=1}^{n} X_i\right)\right)$.

Theorem 3.3: Let $A_1, A_2, \ldots, A_n$ be closed subsets of the closed set *A* such that $\overline{A_0} \subset \overline{A_1} \subset \overline{A_2} \subset \ldots \subset \overline{A_{n-1}} \subset \bar{A}$, then the set *A* is indefinitely extendable from its smallest subset.

Proof: The result follows directly from consequence of Theorem 3.1 or 3.2 which is already established.

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