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On Analytic Review of Hahn–Banach Extension Results with Some Generalizations

Chigozie Emmanuel Eziokwu

Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria

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ABSTRACT

The useful Hahn–Banach theorem in functional analysis has significantly been in use for many years ago. At this point in time, we discover that its domain and range of existence can be extended point wisely so as to secure a wider range of extendibility. In achieving this, we initially reviewed the existing traditional Hahn–Banach extension theorem, before we carefully and successfully used it to generate the finite extension form as in main results of section three.

Key words: Extensions, linear functional, lower bounds, normed space, upper bounds, vector space

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INTRODUCTION AND RESULTS

Introduction

Let X be a linear vector space. A linear operator from X into the space R is called a real linear functional on X. Similarly for X a normed linear space a bounded linear operator from X into R is called a continuous linear functional on X.

Results

The Hahn–Banach theorem is basically defined for R and sometimes holds for a complex linear functional on X when X is a complex space while a complex linear functional on X is obtained when X is a complex space and R is replaced by R.

Theorem 1.2.1 (Hahn–Banach Theorem):[1] Let X be a real vector space, M a subspace of X, and P a real function defined on X satisfying the following conditions:

1. $P(x+y) \leq P(x) + p(y)$.
2. $P(\alpha x) = \alpha p(x) \; \forall x, y \in X$ and positive real α.

Further, suppose that f is a linear functional on M such that $f(x) \leq p(x) \; \forall x \in M$. Then, there exists a linear functional F defined on X for which $F(x) - f(x) \; \forall x \in M$ and $F(x) \leq p(x) \; \forall x \in X$. In other words, there exists an extension F of f having the property of f.

Theorem 1.2.2 (Topological Hahn–Banach Theorem):[2] Let X be a normed space, M a subspace of X, and f a bounded linear functional on M.

1. $F(x) = f(x) \quad \forall x \in M$.
2. $\|F\| = \|f\|$.

Address for correspondence:
Chigozie Emmanuel Eziokwu
E-mail: okereemm@yahoo.com

In other words, there exists an extension F of f which is also bounded linear and preserves the norm. The proof of Theorem 1.2.1 depends on the following lemma:

Lemma 1.2.1:[3] Let X be a vector space and M its proper subspace. For $x_0 \in X - M$, let $N = \{M \bigcup \{x_0\}\}$. Furthermore, suppose that f is a linear functional on M and p a functional on X satisfying the conditions in theorem 1.2.1 such that $f(x) \le p(x)\ \forall x \in M$. Then, there exists a linear functional F defined on N such that $F(x) = f(x)\ \forall x \in M$ and $F(x) \le p(x)\ \forall x \in N$.

In short, this lemma tells us that Theorem 1.2.1 is valid for the subspace generated or spanned by $M \bigcup \{x_0\}$.

Consequences of the Extension Form of the Hahn–Banach Theorem

The proofs of the following important results mainly depend on the proof of Lemma 1.2.1.

Theorem 1.2.3:[4] Let w be a nonzero vector in a normed space X then there exists a continuous linear functional F, defined on the entire space X such that $\| F \| = 1$ and $F(w) = \| w \|$.

Theorem 1.2.4:[5] If X is a normed space such that $F(w) = 0\ \ \forall F \in X^*$, then $w = 0$.

Theorem 1.2.5:[6] Let X be a normed space and M its closed subspace. Further assuming that $w \in X\ \ M(w \in X)$ but $w \in M$. Then, there exists $F \in X^*$ such that $F(m) = 0$ for all $m \in M$, and $F(w) = 1$.

Theorem 1.2.6:[7] Let X be a normed space, M its subspace and $w \in X$ such that $d = \inf \|w - m\| > 0$. It may be observed that this condition is satisfied if m is closed and $w \in (X - M)$. Then, there exists $F \in X^*$ with $\| F \| = 1$, $F(w) \ne 0$, and $F(m) = 0$ for all $m \in M$.

Theorem 1.2.7:[8,9] If X^* is separable, then X is itself separable.

PROOF OF HAHN–BANACH RESULTS

Proof of Lemma 1.2.1[1,9] **due to Siddiqi**

This will help us in developing the proof of theorem 1.2.1 of the Hahn–Banach Theorem. Since $f(x) \le p(x)$ for $x \in M$ and f is linear, we have arbitrary $y_1,\ y_1 \in M$.

$f(y_1, y_2) = f(y_1) f(y_2) \le P(y_1, y_2)$

or

$f(y_1) f(y_2) \le p(y_1 + x_0, y_2 - x_0)$

$\le p(y_1 + x_0) + p(-y_2 - x_0)$

by condition (1) of Theorem 1.2.1.

$$-p(-y_2 + x_0) f(y_2) \leqslant p(y_1 + x_0) f(y_1) \tag{2.1.1}$$

Suppose y_1 is kept fixed and y_2 is allowed to vary over M, then equation (2.1.1) implies that the set of real numbers $\{p(y_1 + x_0) - f(y_2) | y_2 \in M\}$ has upper bounds and hence the least upper bound. Let $\alpha = \sup\{p(y_1 + x_0) - f(y_2) | y_2 \in M\}$. If we keep y_2 fixed and y_1 is allowed to vary over M, equation

(2.1.1) implies that the set of numbers $\left\{p\left(y_2+x_0\right)f\left(y_1\right)\middle| y_1 \in M\right\}$ has lower bounds and hence the greatest lower bound.

Let $\beta = \inf\left\{p\left(y_1+x_0\right)f\left(y_1\right)\middle| y_1 \in M\right\}$. As it is well known that between any two real numbers, there is an always a third real numbers. Let y be a real number such that

$$\alpha \le \gamma \le \beta \tag{2.1.2}$$

It may be observed that if $\alpha = \beta$, then $\gamma = \alpha = \beta$. Therefore, for $\gamma \in M$, we have

$$p(-y-x_0)f(y) \le \gamma \le p(y+x_0) - f(y) \tag{2.1.3}$$

From the definition of N, it is clear that every element x in N can be written as

$$x = y + \lambda x_0 \tag{2.1.4}$$

Where $x_0 \in M$ or $x_0 \in XM$, λ is a uniquely determined real number and γ a uniquely determined vector in M. We now define a real-valued function on N as follows:

$$F\left(x\right) = F\left(y+\lambda x_0\right) = f\left(y\right) + \lambda y \tag{2.1.5}$$

We shall now verify that[8] the well-defined function satisfies the desired conditions, i.e.,

i. F is linear,

ii. $F\left(x\right) = f\left(x\right)\ \forall x \in M$,

iii. $F\left(x\right) = p\left(x\right)\ \forall x \in N$.

iv. F is linear: For

$z_1, z_2 \in N\left(z_1 = y_1 + \lambda_1 x_0, z_2 = y_2 + \lambda_2 x_0\right)$

$F\left(z_1+z_2\right) = F\left(y_1 + \lambda_1 x_0 + y_2 + \lambda_2 x_0\right)$

$= F\left(\left(y_1+y_2\right) + \left(\lambda_1+\lambda_2\right)x_0\right) = f\left(y_1+y_2\right) + \left(\lambda_1+\lambda_2\right)\gamma$

$= f\left(y_1\right) + f\left(y_2\right) + \lambda_1\gamma + \lambda_2\gamma$

as f is linear: Or

$F\left(z_1+z_2\right) = \left[f\left(y_1\right) + \lambda_1\gamma\right] + \left[f\left(y_2\right) + \lambda_2\gamma\right]$

Similarly, we can show that $F\left(\mu z\right) = \mu F\left(z\right)\ \forall z \in N$ and for real μ.

2. If $x \in M$, then γ must be zero in equation (2.1.4) and then equation (2.1.5) gives $F\left(x\right) = f\left(x\right)$
Here, we consider three cases.[9] (See equation 2.1.4)

Case 1, $\lambda = 0$: We have seen that $F\left(x\right) = f\left(x\right)$ and as $f\left(x\right) \le p\left(x\right)$, we get that.

$F\left(x\right) \le p\left(x\right)$

Case 2, $\lambda > 0$: From equation (2.1.3), we have.

$$\gamma \le p\left(y+x_0\right) - f\left(y\right) \tag{2.1.6}$$

Since N is a subspace, $y/\lambda \in N$ replacing y by y/N in equation (2.1.6), we have

$$\gamma \le p(y/\lambda) + x_0 f - \left(f\left(\frac{y}{\lambda}\right)\right)$$

or

$$\gamma \le p\left(\frac{1}{\lambda}(y+\lambda x_0)\right) - f(y/\lambda)$$

By condition (2) of theorem 1.2.1

$$p\left(\frac{1}{\lambda}(y+\lambda x_0)\right) = \frac{1}{\lambda} p(y+\lambda x_0)$$

For $\lambda > 0$ and $f(y/\lambda) = \frac{1}{\lambda} f(y)$ as f is linear. Therefore, $\lambda\gamma \le p(y+\lambda x_0) - f(\gamma)$ or $f(y) + \lambda p(y+\lambda x_0)$. Thus, from equations (2.1.4) and (2.1.5), we have $F(x) \le p(x)\quad \forall x \in N$.

Case 3, $\lambda < 0$: From equation (2.1.3), we have.

$$-p(-y-x_0) - f(y) \le \gamma \tag{2.1.7}$$

Replacing γ by γ/λ in equation (2.1.1), we have

$$-p\left(\frac{-y}{\lambda} - x_0\right) - f\left(\frac{y}{\lambda}\right) \le \gamma$$

or

$$-p\left(\frac{-y}{\lambda} - x_0\right) \le \lambda + f\left(\frac{y}{\lambda}\right) = \gamma + \frac{1}{\lambda} f(y)$$

As f is linear, i.e.,

$$-p\left(\frac{-y}{\lambda} - x_0\right) \le \gamma + \frac{1}{\lambda} f(y) \tag{2.1.8}$$

Multiplying (2.1.8) by λ, we have

$$-\lambda p\left(\frac{-y}{\lambda} - x_0\right) \le \lambda\gamma + f(y)$$

(The inequality in equation (2.1.8) is reversed as λ is negative),

$$(-\lambda) p\left(\left(-\frac{-y}{\lambda}\right)(y+\lambda x_0)\right) \ge F(x) \tag{2.1.9}$$

Since $-\frac{1}{\lambda} > 0$, by condition (2) of Theorem 1.2.1, we have

$$P\left(\left(-\frac{1}{\lambda}\right)(y+\lambda x_0)\right) = -\frac{1}{\lambda} p(y+\lambda x_0) \tag{2.1.10}$$

and so

$$(-\lambda)\left(-\frac{1}{\lambda}\right) p(y+\lambda x_0) \ge F(x)$$

or

$$F(x) \le p(x)\quad \forall x \in N$$

Proof of Theorem 1.2.1.[2,9] due to Siddiqi

Let S be the set of all linear functionals F such that $F(x) = f(x)\ \forall x \in M$ and $F(x) \le p(x)\ \forall x \in X$. That is to say, S is the set of all functionals F extending f and $F(x) \le p(x)$ over X. S is non-empty as not only does F belong to it but there are other functionals also which belong to it by virtue of Lemma 1.2.1, we introduce a relation in S as follows.

For $F_1, F_2 \in S$, we say that F_1 is in relation to F_2 and we write $F_1 < F_2$ if $DF_1 \subset DF_2$ and $F_2 / DF_1 = F_1$ (let DF_1 and DF_2 denote, respectively, the domain of F_1 and $F2 : F_2 / DF_1$ denotes the restriction of F_2 on the domain of F_1. S is a partially ordered set. The relation $<$ is reflexive as $F_1 < F_1$. $<$ is transitive, because for $F_1 < F_2$, $F_2 < F_3$, we have

$DF_1 \subset DF_2$, $DF_2 \subset DF_3$. $F_2 / DF_1 = F_1$ and $F_3 / DF_2 = F_2$, which implies that

$DF_1 \subset DF_3$ and $F_3 / DF_1 = F_1$. $<$ is anti-symmetric. For $F_1 < F_2$;

$DF_1 \subset DF_2$
$F_2 / DF_1 = F_1$

For $F_2 < F_1$;

$DF_2 \subset DF_1$
$F_1 / DF_2 = F_2$
Therefore, we have $F_1 = F_2$.

We now[5] show that every totally ordered subset of S has an upper bound in S. Let $T = \{F_\sigma\}$ be a totally ordered subset of S. Let us consider a functional, say F defined over $\bigcup_\sigma DF_\sigma$. If $x \in \bigcup_\sigma DF_\sigma$, there must be some σ such that $x \in DF_\sigma$, and we define $F(x) = F_\sigma(x)$. F is well defined and its domain $\bigcup_\sigma F_\sigma$ is a subspace of X. $\bigcup_\sigma DF_\sigma$ is a subspace: Let $x, y \in \bigcup_\sigma DF_\sigma$. This implies that $x \in DF_{\sigma_1}$ and $y \in DF_{\sigma_2}$. Since T is totally ordered, either $DF_{\sigma_1} \subset DF_{\sigma_2}$ or $DF_{\sigma_2} \subset DF_{\sigma_1}$. Let $DF_{\sigma_1} \subset DF_{\sigma_2}$. Then, DF_σ, $x \in DF_{\sigma_1}$ which implies that $x \in DF_\sigma\ \forall$ real μ. This shows that DF_σ is a subspace. F is well defined: Suppose $x \in DF_\upsilon$. Then, by the definition of F, we have $F(x) = F_\sigma(x)$ and $F(x) = F_\upsilon(x)$. By the total ordering of T either F_σ extends F_υ or vice-versa and so $F_\sigma(x) = F_\upsilon(x)$ which shows that F is well defined. It is clear from the definition that F is linear, $F(x) = f(x)$ for $x \in D = M$ and $F(x) \le p(x)\ \forall x \in DF$. Thus, for each $F_\sigma < F$; i.e., is an upper bound of T. By Zorn's lemma, there exists a maximal element $\hat{F}$ in S; i.e., $\hat{F}_i$ is a linear extension of $\hat{F}(x) \le p(x)$ and $F < \hat{F}$ for every $F \in S$. The theorem will be proved if we show that $D_{\hat{F}} = X$. We know that $D_{\hat{F}} \subset X$. Suppose there is an element $x \in X$ such that $x_0 \notin D_{\hat{F}} = X$. By lemma 1.1.1, there exists $\hat{F}$ such that $\hat{F}$ is linear, $F(x) = \hat{F}(x) \forall x \in D_{\hat{F}}$, and $\hat{F}(x) \le p(x)$ for $x \in \left[D_F \cup \{x_0\} \right]$ is also an extension of f. This implies that $\hat{F}$ is not maximal element for S which is a contradiction. Hence, $D_F = X$.

Proof of Theorem 1.2.2[3,9] due to Siddiqi

Since f is bounded and linear, we have $\left|f\left(x\right)\right|\leqslant\|f\|\|x\|,\ \ \forall x$. If we define $p\left(x\right)=\|f\|\|x\|$ then $p(x)$ satisfies the conditions of theorem 1.1.1. By theorem 1.2.1, there exists F extending f which is linear and $F(x)\leq p(x)=\| f \| \ \| x \|$ which implies that F is bounded and

$$\|F\|=\|x\|\rightarrow 1^{\sup}F\left(x\right)\leqslant\|f\|$$

On the other hand,[9] for $x\in M$, $\left|f\left(x\right)\right|\leqslant\|F\|$. Hence, $\| f \|=\| F \|$.

Remark 2.2.1: The Hahn–Banach theorem is also valid for normed spaces defined over the complex field.

Consequences of the Extension Form of the Hahn–Banach Theorem

The proofs of the following important results mainly depend on theorem 1.2.2.

Proof of Theorem 1.2.3.*[4,9] *due to Siddiqi

Let $M=[w=m/m=\lambda w,\in R]$ and $f:M\Rightarrow R$ such that $f(m)=\lambda \| w \|$.

f is linear

$$[f(m_1+m_2)=(\lambda_1+\lambda_2)\| w \|]$$

where $m_1=\lambda_1 w$ and $m_2=\lambda_2 w$ or

$$f\left(m_1+m_2\right)=\left(\lambda_1+\lambda_2\right)\|w\|=\lambda_1\|w\|+\lambda_2\|w\|=f\left(m_1\right)+f\left(m_2\right)$$

Similarly, $f\left(\mu m\right)=\mu f\left(m\right)\quad \forall\mu\in R$. f is bounded $\left(\left|f\left(m\right)\right|\right)=\|\lambda w\|=\|m\|$ and so $\left|f\left(m\right)\right|\leq k\|m\|$ where $\left(0\leq k\leq 1\right)$ and

$$f\left(w\right)=\|w\|\ \left(\text{if } m=w,\ \text{ then } \lambda=1\right)$$

By theorem 1.2.2,

$$\|f\|=_{\substack{m\in M\\ \|m\|=1}}^{\sup}\left|f\left(m\right)\right|=_{\|m\|=1}^{\sup}\left|\lambda\right|\|w\|=_{\|m\|=1}^{\sup}\|m\|=1$$

Since f, defined on M, is linear and bounded (and hence continuous) and satisfies the conditions $f\left(w\right)=\|w\|$ and $\|f\|=1$; by Theorem 1.2.2, there exists a continuous linear functional F over X extending f such that $\|F\|=1$ and $F\left(w\right)=\|w\|$.

Proof of Theorem 1.2.4.*[5,9] *due to Siddiqi

Suppose $w\neq 0$ but $F\left(w\right)=0$ for all $F\in X^*$. Since $w\neq 0$, by theorem 1.2.1., by theorem 1.2.3, there exists a functional $F\in X^*$ such that $\| F \|=1$ and $F(w)=\| w \|$. This shows that $F\left(w\right)\neq 0$ which contradiction is. Hence, if $F\left(w\right)=0\ \forall F\in X^*$, then w must be zero.

Proof of Theorem 1.2.5.*[6,9] *due to Siddiqi

Let $w\in XM$ and $d=_{m\in M}^{\inf}\|w-m\|$. Since M is a closed subspace and $M,d>0$. Suppose N is the subspace spanned by w and M; i.e.,$n\in N$ if and only if

$N = \lambda w + m, \lambda \in R, m \in M$ (2.1.11)

Define a functional on N as follows:

$F(n) = \lambda$

F is linear and bounded: $f(n_1 + n_2) = \lambda_1 + \lambda_2$, where $n_1 = \lambda_1 w + m$ and $n_2 = \lambda_2 w + m$. Hence, $f(n_1 + n_2) = f(n_1) + f(n_2)$. Similarly, $f(\mu n) = \mu f$ for real μ. Thus, f is linear. To show that f is bounded, we need to show that there exists $K > 0$ such that $|f(\mathrm{n})| \le \|n\| \ \forall n \in N$. We have

$\|n\| = \|m + \lambda w\| = \left\| -\lambda\left(-\frac{m}{\lambda} - w\right)\right\| = |\lambda| \left\| -\frac{m}{\lambda} - w\right\|$

Since $-m\lambda \in M$ and $d = \inf_{m \in M} \|w - m\|$, we see that $\left\| -\frac{m}{\lambda} - w\right\| \ge d$. Hence, $\|n\| \ge |\lambda|$or $|-\lambda| \le \|n\| / d$ By definition, $|f(n)| = |\lambda| \le \|n\| / d$ or $|f(n)| = k$ where $k \ge \frac{1}{d} > 0$. Thus, f is bounded. $N = w$implies that $\lambda = 1$ and therefore, $f(w) = 1$. $N = m \in M$ implies that $\lambda = 1$ and therefore, from the definition of f, $f(m) = 0$. Thus, f is bounded linear and satisfies the conditions $f(w) = 1$ and $f(m) = 0$. Hence, by theorem 1.2.2, there exists F defined over X such that F is an extension of f and F is bounded linear, i.e., $F \in X^*, F(w) = 1$ and $F(m) = 0 \ \forall m \in M$.

Proof of Theorem 1.2.6.[7,9] ***due to Siddiqi***

Let N be the subspace spanned by M and (see equation (2.1.11)). Define f on N as $f(n) = \lambda d$, proceeding exactly as in the proof of theorem 1.2.5, we can show that f is linear and bounded on N, $|f(n)| = |\lambda| d \le \|n\|$, $f(w) = d \ne 0$, and $f(m) = 0$ for all $m \in M$ since $|f(n)| \le \|n\|$, we have

$\|f\| \le 1$ (2.1.12)

For arbitrary $\in > 0$, by the definition of d, there must exist an $m \in M$ such that $\|w - m\| < d + \in$ Let $z = \frac{w - m}{\|w - m\|}$. Then, $\|z\| = \frac{w - m}{\|w - m\|} = 1$ and $f(z) = f(w - m) = d / \|w - m\|$. By definition, $f(n) = \lambda d$; $n = \|w - m\|$, then $\lambda = 1$; and so $f(w - m) = d$;

$f(z) > \frac{d}{d + \in}$ (2.1.13)

By theorem 1.2.2. $\|f\| = \sup_{\|m\|=1} |f(m)|$. Since $\|z\| = 1$, equation (2.1.13) implies that $f(z) > \frac{d}{d + \in}$. Since $\in > 0$ is arbitrary, we have

$\|f\| \ge 1$ (2.1.14)

From equations (2.1.12) and (2.1.14) have $\|f\| = 1$. Thus, f is bounded and linear, $f(m) = 0 \ \forall m \in M$; $f(w) \ne 0$ and $\|f\| = 1$. By theorem 1.2.2, there exists $F \in X^*$ such that $F(w) \ne 0$; $F(m) = 0$ for all $m \in M$ and $\|f\| = 1$.

Proof of Theorem 1.2.7.*[8,9] *due to Siddiqi

Let $\{f_n\}$ be a sequence in the surface of the unit sphere S of

$X^*\left[S=\{F\in X^*/\|F\|=1\}\right]$

such that $[F_1,F_2,\ldots,F_n]$ is a dense subset of S. By theorem 1.2.2,

$\|F\|=\sup_{\|v\|=1}=|F(v)|$

and so for $\in>0$, there exists $v\in X$ such that $\|v\|=1$ and

$$(1-\in)\|F\|\le|F(v)| \tag{2.1.15}$$

Putting $\in=\frac{1}{2}$ in equation (2.1.15), there exists $v\in X$ such that $\|v\|=1$ and $\frac{1}{2}\|F\|\le|F(v)|$.

Let $\{v_n\}$ be a sequence such that $\|v_n\|=1$; $\frac{1}{2}\|F_n\|\le|F_n(v_n)|$; and M be a subspace spanned by $\{v_n\}$. Then, M is separable by its construction. In other to prove that X is separable, we show that $M=X$ suppose $X\neq M$; then, there exists $w\in X; w\notin M$ by theorem 1.2.2, there exists $F\in X^*$such that $\|F\|=1$

$$F(w)\neq 0 \tag{2.1.16}$$

and $F(m)=0\ \forall m\in M$. In particular, $F(v_n)=0\ \forall n$, where

$$\frac{1}{2}F_n\le|F_n(v_n)|=|F_n(v_n)-F(v_n)+F(v_n)|\le|F_nv_n-F(v_n)|+|F(v_n)|$$

Since $\|v_n\|=1$ and

$F(v_n)=0\ \forall n$

We have

$$\frac{1}{2}\|F_n\|\le|F_n-F| \tag{2.1.17}$$

We can choose $\{F_n\}$ such that

$$\lim_{n\to\infty}\|(F_n-F)\|=0 \tag{2.1.18}$$

Because $\{F_n\}$ is a dense subset of S. This implies from equation (2.1.17) that $\|F_n\|=0\ \forall n$.

Thus, using equations (2.1.16), (2.1.18), we have

$$I=\|F\|=\|F-F_n+F_n\|\le\|F-F_n\|+\|F_n\|\le\|F-F_n\|+2\|F-F_n\|$$

or

$$1=\|F\|=0,$$

which is contradiction. Hence, our assumption is false and $X=M$.

MAIN RESULTS ON THE GENERALIZED HAHN–BANACH THEOREM

Theorem 3.1: Let X be a real vector space, $M-$ a subspace of X, and P_i a sequence of real function s defined on X satisfying the following conditions:

i. $P_i\left(\sum_{i=1}^{n}(x_i)\right)\le\sum_{i=1}^{n}p_ix_i$

ii. $P_i(\alpha_i x_i) = \alpha_i p_i(x_i)$

For each $x_i \in X$ and α_i all positive.

Further, suppose that f_i is a sequence of linear functional on M such that

$$\sum_{i=1}^{n} f_i(x_i) \le \sum_{i=1}^{n} p_i(x_i) \ \forall x_i \in M$$

Then, there exists sequence of linear functional F_i defined on X for which

$$\sum_{i=1}^{n} F_i(x_i) = \sum_{i=1}^{n} f_i(x_i) \ \ \forall x_i \in M$$

and

$$\sum_{i=1}^{n} F_i(x_i) \le \sum_{i=1}^{n} p_i(x_i) \ \ \forall x_i \in X$$

In other words, there exists sequence of extensions F_i of F having the property of F_i.

Proof: The statement and proof of the following Lemma will be very significant in the proof of the Generalized Hahn–Banach theorem.

Lemma: Let X be a vector space and μ its proper subspace. For each $x_i \in X - M$, let $N = \left[m \bigcup \{x_i\}\right]$.

Furthermore, suppose that f_i is a sequence of linear functionals on M and $p_i - \alpha$ sequence of functionals on X satisfying the conditions of theorem 3.1 such that

$$\sum_{i=1}^{n} f_i(x_i) \le \sum_{i=1}^{n} p_i(x_i) \ \ \forall x_i \in M$$

Then, there exists a sequence of linear functional F_i defined on N such that

$$\sum_{i=1}^{n} F_i(x_i) = \sum_{i=1}^{n} f_i(x_i), \ \forall x_i \in M$$

and

$$\sum_{i=1}^{n} F_i(x_i) \le \sum_{i=1}^{n} p_i(x_i), \ \forall x_i \in N$$

This Lemma implies that theorem 3.1 is valid for the subspace generated or spanned by $M - \{x_i\}$.

Proof: Since

$$\sum_{i=1}^{n} f_i(x_i) \le \sum_{i=1}^{n} p_i(x_i)$$

For $x_i \in M$ and $\sum_{i=1}^{n} f_i$ are linear, we have for arbitrary $y_i \in P$

$$\sum_{i=1}^{n} f_i \Delta y_i = \sum_{i=1}^{n} \Delta f_i(y_i) \le \sum_{i=1}^{n} p_i \Delta y_i$$

or

$$\sum_{i=1}^{n} \Delta f_i(y_i) \le \sum_{i=1}^{n} p_i \Delta(y_i + x_0) \le \sum_{\substack{i=1 \\ i \ne even}}^{n} p_i(y_i + x_0) + \sum_{\substack{i=1 \\ i \ne odd}}^{n} p_i(-y_{i+1} - x_0)$$

By condition 1 of theorem 1.2.1., thus by regrouping the terms of γ_{i+1} on one side and those of γ_i on the other side, we have

$$\sum_{i=1}^{n}\left[-p_i\left(y_{i+1}-x_0\right)-f_i\left(y_{i+1}\right)\right]\leq\sum_{i=1}^{n}\left[p_i\left(y_i+x_0\right)\right]-f_i\left(y_{i+1}\right) \quad (3.1)$$

Suppose $y_i's$ are kept fixed and $y_{i+1}'s$ are allowed to vary over M, then equation (3.1) implies that the set of real number $\left\{p_i\left(y_i+x_0\right)\right\}-f_i\left(y_i\right)\ \ y_i\in M$ has lower bounds and hence greatest lower bound by Remark 1.1.

Let

$$R=\inf\left\{p_i\left(y_i+x_0\right)-f_i\left(y_i\right): y_i\in M\right\}$$

From equation (3.1), it is clear that $\alpha\leq\beta$. As it is well known that between any two real numbers, there is always a third real number. Let p be a real number such that

$$\alpha<\gamma=\beta \quad (3.2)$$

It may be observed that if $\alpha=\beta$, then $\gamma=\alpha=\beta$. Therefore, for all $y\in M$, we have

$$\sum_{i=1}^{n}f\left[\left(-p_i\left(-y_i-x_0\right)-f_i\left(y_i\right)\right)\right]\leq\gamma\leq\sum_{i=1}^{n}\left[\left(p_i\left(y_i+x_0\right)-f_iy_i\right)\right] \quad (3.3)$$

From the definition of N, it is clear that every element x_i in N can be written as

$$x_i=y_i+\lambda x_0 \quad (3.4)$$

Where $x_0\in M$ or $x_0\in X-M$, λ is uniquely determined real number and γ is uniquely determined vector in M. We now define a sequence of real valued functions on N as follows

$$F_ix_i=F_i\left(y_i+\lambda x_0\right)=f_i\left(y_i\right)+\lambda \quad (3.5)$$

Where y is given by equation (3.2) and x is as in equation (3.4). We shall now verify that the well-defined sequence of functions $F_i\left(x_i\right)$ satisfies the desired conditions, i.e.,

i. $\sum_{i=1}^{n}F_i$ is linear

ii. $\sum_{i=1}^{n}f_i\left(x_i\right)=\sum_{i=1}^{n}F_i\left(x_i\right)\ \ \forall x_i\in M$

iii. $\sum_{i=1}^{n}F_i\left(x_i\right)\leq\sum_{i=1}^{n}p_i\left(x_i\right)\ \ \forall x_i\in N$

1. $\sum F_i$ is linear

For

$z_1,z_2,\ldots,z_n\in N\left(z_1=y_1+\lambda_1x_0,z_2=y_2+\lambda_2x_0,\ldots,z_n=y_n+\lambda_nx_0\right),$

$F_1\left(y_1+\lambda_1x_0+y_2+\lambda_2x_0+\cdots+y_n+\lambda_nx_0\right)$

$=f_1\left(y_1+y_2+\ldots+y_n\right)+\left(\lambda_1+\lambda_2+\ldots+\lambda_n\right)x_0$

$=f_i\left(y_1+y_2+\ldots+y_n\right)+\left(\lambda_1+\lambda_2+\ldots+\lambda_n\right)\gamma$

$=f_1\left(y_1\right)+f_2\left(y_2\right)+\ldots+f_n\left(y_n\right)+\lambda_1\gamma+\lambda_2\gamma+\ldots+\lambda_n\gamma$

as f_i is linear

or

$$F_i\left(z_1+z_2+\ldots+z_n\right)=\left[f_1\left(y_1\right)+\lambda_1\gamma\right]+\left[f_2\left(y_2\right)+\lambda_2\gamma\right]+\left[f_n\left(y_n\right)+\lambda_n\gamma\right]$$

$$=F_1\left(z_1\right)+F_2\left(z_2\right)+\ldots+F_n\left(z_n\right)$$

Similarly,

2. $\sum_{i=1}^{n} f\left(\alpha z\right)=\alpha\sum_{i=1}^{n} F\left(Z\right)$ for each $z\in N$ and for real α

3. If $x_1\in M$, then λ_i must be zero in equation (3.4).

Case 1: $\lambda_i=0$:} We have seen that

$$\sum_{i=1}^{n} F_i\left(x_i\right)=\sum_{i=1}^{n} f_i\left(x_i\right)$$

and as

$$\sum_{i=1}^{n} f_i\left(x_i\right)\le\sum_{i=1}^{n} p_i\left(x_i\right)$$

we get that

$$\sum_{i=1}^{n} F_i\left(x_i\right)\le\sum_{i=1}^{n} P_i\left(x_i\right)$$

Case 2: $\lambda>0$: From equation (3.2), we have

$$\gamma\le\sum_{i=1}^{n} p_i\left(y_i+x_0\right)-f\left(y_i\right) \tag{3.6}$$

since N is a subspace, $y_i\lambda\in N$ replacing y by y/λ in equation (3.6), we have

$$\gamma\le\sum_{i=1}^{n}\left[p_i\left(\frac{y_i}{\lambda_i}+x_0\right)-f_i\left(\frac{y_i}{\lambda_i}\right)\right]$$

or

$$\gamma\le\sum_{i=1}^{n}\left[p_i\left(\frac{1}{\lambda_1}\right)\left(y_i+\lambda_i x_0\right)-f_i\left(\frac{y_i}{\lambda_i}\right)\right]$$

By condition (2) of theorem (2.1),

$$\sum_{i=1}^{n} p_i\left[\frac{1}{\lambda_i}\left(y_i+x_0\right)\right]=\sum_{i=1}^{n}\frac{1}{\lambda_i}\left[p_i\left(y_i+\lambda x_0\right)\right]$$

For $\lambda>0$ and $f_i\left(\frac{y_i}{x_i}\right)=\frac{1}{\lambda_i}f_i\left(y_i\right)$ as f_i is linear.

Therefore,

$$\sum_{i=1}^{n}\lambda_i\gamma\le\sum_{i=1}^{n}\left[p_i\left(y_i+\lambda_i x_0\right)\right]=f_i\left(y_i\right)$$

or

$$\sum_{i=1}^{n} f_i\left(y_i\right)+\lambda_i y_i\le\sum_{i=1}^{n} p_i\left(y_i+\lambda x_0\right)$$

Thus, from equations (3.4) and (3.5), we have

$$\sum_{i=1}^{n}\left[p_i\left(-y_i-x_0\right)-f_i\left(y_i\right)\right]\le\sum_{i=1}^{n}\gamma_i \tag{3.7}$$

Replacing γ_i by $\frac{y_i}{\lambda}$ in equation (3.7), we have

$$\sum_{i=1}^{n}\left[p_i\left(\frac{-y_i}{x_i}-x_0\right)-f_i\frac{y_i}{x_i}\right]\le\sum_{i=1}^{n}\gamma_i$$

or

$$\sum_{i=1}^{n}\left[p_i\left(\frac{-y_i}{x_i}-x_0\right)\right]\le\sum_{i=1}^{n}\left[\gamma_i+f_i\left(\frac{y_i}{x_i}\right)\right]=\sum_{i=1}^{n}\left[\gamma_i+\frac{1}{\lambda}f_i\left(y_i\right)\right]$$

As f_i is linear,

$$\sum_{i=1}^{n}\left[p_i\left(\frac{y_i}{x_i}-x_0\right)\right]\le\sum_{i=1}^{n}\left[\gamma_i+\frac{1}{\lambda}f_i\left(y_i\right)\right] \tag{3.8}$$

Multiplying (3.8) by λ, we have

$$\sum_{i=1}^{n}\left[\left(\lambda_i p_i\right)\left(\frac{-y_i}{x_i}-x_0\right)\right]\ge\sum_{i=1}^{n}\left[\lambda_i\gamma_i+f_i\left(y_i\right)\right]$$

(The inequality in (3.8) is reversed as λ is negative) or

$$\sum_{i=1}^{n}\left[\left(-\lambda_i\right)p_i\left(-\frac{1}{\lambda_i}\right)\left(y_i+\lambda_i x_0\right)\right]\ge\sum_{i=1}^{n}F_i\left(x_i\right)$$

Since $-\frac{1}{\lambda_i}>0$, by theorem 2.1, we have

$$\sum_{i=1}^{n}\left[p_i\left(-\frac{1}{\lambda_i}\right)\left(y_i+\lambda_i x_0\right)\le\sum_{i=1}^{n}\left(-\frac{1}{\lambda_i}\right)p_i\left(y_i+\lambda x_0\right)\right]$$

and so

$$\sum_{i=1}^{n}\left[\left(-\lambda_i\right)\left(-\frac{1}{\lambda_i}\right)p_i\left(y_i+\lambda_i x_0\right)\right]\ge\sum_{i=1}^{n}F_i\left(x_i\right)$$

or

$$\sum_{i=1}^{n}F_i\left(x_i\right)\le\sum_{i=1}^{n}p_i\left(x_i\right)\ \ \forall x_i\in N$$

and hence, the proof.

Now, having established the proof of the above stated lemma, we then make its use in the proof of theorem 3.1 earlier stated. Hence: Let S be the set of sequence of all linear functional F_i such that

$$\sum_{i=1}^{n}F_i\left(x_i\right)=\sum_{i=1}^{n}f_i\left(x_i\right)\ \ \forall x_i\in M$$

and

$$\sum_{i=1}^{n}F_i\left(x_i\right)=\sum_{i=1}^{n}p_i\left(x_i\right)\ \ \forall x_i\in X$$

That is to say that S is the set of sequences of all functional F_i extending f_i and $\sum_{i=1}^{n} F_i\left(x_i\right)=\sum_{i=1}^{n} p_i\left(x_i\right)$ over X; S is a non-empty as not only does F_i belong to it but there are other functional also which belong to it by virtue of theorem (1.2.1), we introduce a relation which is as follows.

For $F_i, F_{i+1} \in S$, we say that F_i is in relation to F_{i+1} and we write $F_i < F_{i+1}$. If $DF_i \subset DF_{i+1}$ and $F_{i+1} / DF_i = F_i$, let DF_i and DF_{i+1} denote, respectively, the domain of F_i and F_{i+1}. F_{i+1} / DF_i on the partially ordered set. The relation $<$ is reflexive as $F_i < F_{i+1}$: $<$ is transitive because for $F_i < F_{i+1}$; $F_{i+1} < F_{i+2}$, we have $DF_i \subset DF_{i+1}$; $DF_{i+1} \subset DF_{i+2}$. $F_{i+1} / DF_i = F_i$; and $F_{i+2} / DF_{i+1} = F_{i+1}$, which implies that $DF_i \subset DF_3$ and $F_{i+2} / DF_i = F_i$: $<$ is anti-symmetric for $F_i < F_{i+1}$.

$DF_i \subset DF_{i+1}$

$F_{i+1} / \ DF_i = F_i$

for

$F_{i+1} < F_i$

$DF_{i+1} \subset DF_i$

$F_i / DF_{i+1} = F_{i+1}$

Therefore, we have $F_i = F_{i+1}$. We now show that every totally ordered subject of S has an upper bound in S Let $T=\left\{F_{\sigma_1}\right\}$ be a sequence of totally ordered subset of S. Let us consider a sequences of functional say F defined over $\prod_{\sigma_1} DF_{\sigma_1}$.

If $x \in \prod_{\sigma_1} DF_{\sigma_1}$, there must be some σ_i such that $x_i \in DF_{\sigma_i}$ and we defined $F_i\left\{x_i\right\}=F_{\sigma_i}\left(x_i\right)$. F is well defined and its domain $\prod_{\sigma_i} DF_{\sigma_i}$ is a subspace of X. $\prod_{\sigma_i} DF_{\sigma_i}$ is a subspace. Let $x_1, x_2, \ldots, x_n \in \prod_{\sigma_i} DF_{\sigma_i}$. This implies that $x_i \in \prod_{\sigma_i} DF_{\sigma_i}$ and $x_{i+1} \in \prod_{\sigma_i} DF_{\sigma_{i+1}}$.

Since T is totally ordered, either $DF_{\sigma_i} \subset DF_{\sigma_{i+1}}$ or $DF_{\sigma_{i+1}} \subset DF_{\sigma_i}$. Let $DF_{\sigma_i} \subset DF_{\sigma_{i+1}}$. Then, $x \in DF_{\sigma_{i+1}}$ and so

$x_i + x_{i+1} \in DF_{\sigma_{i+1}}$

or

$x_i + x_{i+1} \in \prod_{\sigma_i} DF_{\sigma_i}$

Let $x_i \in DF_{\sigma_i}$ implies that $\vartheta x \in \prod_{\sigma_i} DF_{\sigma_i}$ $\forall$ real ϑ. This shows that $\prod_{\sigma_i} DF_{\sigma_i}$ is a subspace. F is well defined: Suppose $x_i \in DF_{\sigma_i}$ and $x_i \in DF_{\vartheta_i}$. Then, by the definition of F_i, we have $F_i\left(x_i\right)=F_{\sigma_i}\left(x_i\right)$ and $F_i\left(x_i\right)=F_{\vartheta_i}\left(x_i\right)$. By the total ordering of T, either F_{σ_i} and extend F_{ϑ_i} or vice-versa and so $F_{\sigma_i}\left(x_i\right)=F_{\vartheta_i}\left(x_i\right)$ which shows that F_i is well defined. It is clear from the definition that F_i is linear,

$$\sum_{i=1}^{n} F_i\left(x_i\right) \leq \sum_{i=1}^{n} f_i\left(x_i\right) \ \forall x_i \in D_f = M$$

and

$$\sum_{i=1}^{n} F_i\left(x_i\right) \leq \sum_{i=1}^{n} p_i\left(x_i\right) \ \forall x_i \in D_f$$

Thus, for each $F_{\sigma_i} \in T, F_{\sigma_i} < F_i$, i.e., is an upper bound of T. By Zorn's lemma, there exists a maximal element $\bar{F}_i$ in S, i.e., $\bar{F}_i$ is a linear extension of

$$f_i \cdot \sum_{i=1}^{n} \bar{F}_i(x_i) \le \sum_{i=1}^{n} p_i(x_i)$$

and

$$\sum_{i=1}^{n} F_i < \sum_{i=1}^{n} \bar{F}_i \text{ for every } F_i \in S$$

The theorem will be proved if we show that $D_{\bar{F}_i} = X$. We know that $D_{\bar{F}_i} \subset X$. Suppose there is an element $x \in X$ such that $x_0 \notin D_{\bar{F}_i}$. By the above lemma 3.1, there exists $\bar{F}_i$ such that $\bar{F}_i$ is linear,

$$F_i(x_i) = \bar{F}_i(x_i) \ \forall x_i \in D_{\bar{F}_i}$$

and

$$\sum_{i=1}^{n} \bar{F}_i(x_i) \le \sum_{i=1}^{n} p_i(x_i) \text{ for } x_i \in \left[D_F V\{x_0\}\right]$$

Is also an extension of f. This implies that F is not maximal element S which is a contradiction. Hence, $D_{f_i} = X$. Hence, the proof.

Theorem 3.2 (on the generalized form of the topological Hahn–Banach theorem): Let x be a normed space $M-$ a subspace of X and f_i- a sequence of bounded linear functional of M, then there exist a sequence of bounded functional F_i on x such that

$$\sum_{i=1}^{n} F_i(x_i) = \sum_{i=1}^{n} f_i(x_i) \ \forall x_i \in M$$

$$\left\|\sum_{i=1}^{n} F_i\right\| = \left\|\sum_{i=1}^{n} f_i\right\|$$

Proof 3.2: Since f_i is bounded and linear, we have

$$\left\|\sum_{i=1}^{n} f_i(x_i)\right\| \le \left\|\sum_{i=1}^{n} f_i\right\| \left\|\sum_{i=1}^{n} x_i\right\| \ \forall x_i$$

If we have defined $\sum_{i=1}^{n} p_i(x_i) = \left\|\sum_{i=1}^{n} f_i\right\| \left\|\sum_{i=1}^{n} x_i\right\|$ then $\sum_{i=1}^{n} p_i(x_i)$ satisfies the conditions of the theorem (3.1) and by this theorem, there exists $\sum_{i=1}^{n} F_i$ extending $\sum_{i=1}^{n} f_i$ which is linear and $\sum_{i=1}^{n} F_i(x_i) \le \sum_{i=1}^{n} p_i(x_i) \ \ \forall x_i \in X$, we have $\sum_{i=1}^{n} -F_i(x_i) = \sum_{i=1}^{n} F_i(x_i)$ as F_i is linear and so by the above relation

$$\sum_{i=1}^{n} F_i(x_i) \le \sum_{i=1}^{n} p_i(-x_i) = \left\|\sum f_i\right\| \left\|\sum -x_i\right\| = \left\|\sum f_i\right\| \left\|\sum x_i\right\| = \sum_{i=1}^{n} p_i(x_i)$$

Thus,

$$\sum_{i=1}^{n} F_i(x_i) \le \sum_{i=1}^{n} p_i(x_i) = \left\|\sum f_i\right\| \left\|\sum x_i\right\|$$

Which implies that F_i is bounded and

$$\left\|\sum F_i\right\| = \sup_{\left\|\sum x_i\right\|=1} \left\|\sum F_i(x_i)\right\| \le \left\|\sum f_i\right\| \quad (3.9)$$

On the other hand, for $x \in M$,

$$\left\|\sum_{i=1}^{n} f_i\left(x_i\right)\right\| = \left\|\sum_{i=1}^{n} F_i\left(x_i\right)\right\| \leq \left\|\sum_{i=1}^{n} x_i\right\|$$

and so

$$\left\|\sum_{i=1}^{n} f_i\right\| = \sup_{\|x\|=1} \left\|\sum f\left(x\right)\right\|$$

$$\sum_{i=1}^{n} f_i = \sup_{\|x\|=1} \left\|\sum_{i=1}^{n} f_i\left(x_i\right)\right\| \leq \left\|\sum_{i=1}^{n} F_i\right\| \tag{3.10}$$

Hence, by (3.9) and (3.10), we have

$$\left\|\sum_{i=1}^{n} f_i\right\| = \left\|\sum_{i=1}^{n} F_i\right\|$$

Proof of theorem 3.3: Let $M = \left[\left\{w_i\right\}\right] = \left\{m_i : m_i = \lambda_i \in R\right\}$ and $F_i : M \rightarrow R$ such that

$$\sum_{i=1}^{n} f_i\left(m_i\right) = \sum_{i=1}^{n} \lambda_i \left\|w_i\right\|$$

f_i is a linear since

$$\sum_{i=1}^{n} f_i\left(m_i + m_j\right) = \sum_{i=1}^{n}\left(\lambda_i + \lambda_j\right) w_i$$

where $m_i = \lambda_i w_i$ and $m_j = \lambda_j w_i$ or

$$\sum_{i=1}^{n} f_i\left(m_i + m_j\right) + \sum_{i=1}^{n} \lambda_i w_i + \sum\nolimits^{n}_{\substack{i=1 \\ j=i+1}} \lambda_i w_i = \sum_{i=1}^{n} f_i m_i + \sum\nolimits^{n}_{\substack{i=1 \\ j=i+1}} f_i m_j$$

We now state the rest of the generalized results without their proofs as they directly follow.

Theorem 3.4: Let w_i be a sequence of non-zero vectors in a normed space X. Then, there exists a sequence of continuous linear functional F_i defined on the entire space X such that $\left\|F_i\right\| = 1$ and

$$\sum_{i=1}^{n} F_i\left(w_i\right) = \sum_{i=1}^{n} w_i$$

Theorem 3.5: If X is a normed space such that $\sum_{i=1}^{n} F_i\left(w_i\right) = 0 \ \forall F_i \in X^*$. Then, $\sum_{i=1}^{n} w_i = 0$.

Theorem 3.6: Let X be a normed space and M its closed subspace. Further assume that $w_i \in XM$. Then, there exists $F_i \in X^*$ such that $F_i\left(m_i\right) = 0$ for all $m_i \in M$ and $F_i\left(w_i\right) = 1$.

Theorem 3.7: Let X be a normed space, m its subspace and $w_i \in X$ such that $d = \sum_{m_i \in M} \inf \left\|w_i - m_i\right\| > 0$.

Theorem 3.8: If $\bigcup_{i=1} x_i^*$ is separable, then $\bigcup_{i=1} X_i$ is itself separable.

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