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RESEARCH ARTICLE

The Introduction of Extrapolated Block Adams Moulton Methods for Solving First-order Delay Differential Equations

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ABSTRACT

In this paper, the discrete schemes of extrapolated block Adams Moulton methods were obtained through the continuous formulation of the linear multistep collocation method by matrix inversion approach for the numerical solutions of first-order delay differential equations (DDEs) without the use of interpolation techniques in evaluating the delay term. The delay term was computed by a valid idea of sequence. The advantages, convergence, stability analysis, and central processing unit time at a constant step size b of the proposed method over other existing methods are pointed out.

Key words: Adams Moulton method, block method, extended future points, first-order delay differential equations

INTRODUCTION

The rising of mathematical problems in real-life situations has drawn so much attention in all disciplines and needs to be address properly. Most scholars have demonstrated the application of numerical methods in the solution of delay differential equations (DDEs) in the field of engineering, physics, medicine, and economics using interpolation techniques such as Hermite, Nordsieck, and Newton divided difference and Neville's interpolation in evaluating the delay term as studied by Tziperman *et al.*, Bocharov *et al.*, Oberle, and Pesh, Evans and Raslan, Seong and Majid.[1-6] Real-life situations have shown that delays can be seen everywhere and ignoring it means ignoring reality because the solution of DDEs takes into account the current state and the history part of the system being modeled whereas the evolution of ODEs depends only on the current state. The DDEs are differential equations in which the derivatives of the unknown function at a certain time are given in terms of the values of the function at previous times. One of the hindrances encountered by these scholars in the use of interpolation techniques to evaluate the delay term of DDEs was studied by Majid *et al.*[7] that the computational method use in solving DDEs should be at least the same with the order of the interpolating polynomials which is very hard to achieve; otherwise, the accuracy of the method will not be preserved. Therefore, it is required that in the evaluation of the delay term, using an accurate and efficient formula should be considered.

To overcome the hindrance posed using interpolation techniques in checking the delay term, we shall apply the valid expression of the sequence formulated by Sirisena and Yakubu[8] and incorporate it into the first-order DDEs before its numerical evaluation. This approach has been successfully applied by Osu *et al.*, Chibuisi *et al.*[9-11] in finding the numerical solution of first-order DDEs without the application of the interpolation techniques in evaluating the delay term.

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In this paper, we shall formulate and apply extrapolated block Adams Moulton methods in solving some first-order DDEs as developed by Ballen and Zennaro.[12]

$$y'(t) = f(t, y(t), y(t-\tau)),\ \text{for } t?t_0,\ \tau>0$$

$y(t)=m(t)$, for $t \leq t_0$ (1)

Where, $m(t)$ is the initial function, τ is called the delay, $(t-\tau)$ is called the delay argument, and $y(t-\tau)$ is the solution of the delay term. The results obtained after the application of the proposed method shall be compared to other existing methods studied by Sirisena and Yakubu, Osu *et al*.[8,9,13] to prove its advantage.

DEVELOPMENT OF LINEAR MULTISTEP COLLOCATION PROCEDURE

The k-step linear multistep collocation procedure with m collocation points was derived in Ballen and Zennar[12] as;

$$y(x) = \sum_{a=0}^{e-1} \alpha_a(x) y_{z+a} + b\sum_{a=0}^{w-1} \beta_a(x) f_{z+a}(x, y(x)) \quad (2)$$

From Equation (2), the continuous expression of extended block Adams Moulton methods can be expressed as

$$y(x) = \sum_{a=0}^{e-1} \begin{matrix} \alpha_a(x) y_{z+a} + b\sum_{a=0}^{w-1} \beta_a(x) f_{z+a}(x, y(x)) + \\ b\sum_{a=0}^{w-1} \gamma_a(x) g_{z+a}(x, y(x)) \end{matrix} \quad (3)$$

Where, $\alpha_a(x)$, $\beta_a(x)$ and $\gamma_a(x)$ are continuous coefficients of the technique defined as

$$\alpha_a(x) = \sum_{n=0}^{e+w-1} \alpha_{a,n+1} x^n \text{ for } a = \{0, 1\ldots,e-1\} \quad (4)$$

$$b\beta_a(x) = \sum_{n=0}^{e+w-1} b\beta_{a,n+1} x^n \text{ for } a = \{0, 1\ldots,w-1\} \quad (5)$$

$$b\gamma_a(x) = \sum_{n=0}^{e+w-1} b\gamma_{a,n+1} x^n \text{ for } a = \{0, 1\ldots,w-1\} \quad (6)$$

Where, $a = \{0, 1\ldots,w-1\}$ are the w collocation points, x_{z+a}, $a = 0, 1\ldots,e-1$ are the e arbitrarily chosen interpolation points, and b is the constant step size.

To get $\alpha_a(x)$, $\beta_a(x)$ and $\gamma_a(x)$,[13] formulated a matrix equation of the form

$EH=I$ (7)

Where, I is the square matrix of dimension $(e+w)\times(e+w)$ while E and H are matrices defined as

$$E = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{e-1,1} & b\beta_{0,1} & \cdots & b\beta_{w-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{e-1,2} & b\beta_{0,2} & \cdots & b\beta_{w-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,e+w} & \alpha_{1,e+w} & \cdots & \alpha_{e-1,e+w} & b\beta_{0,e+w} & \cdots & b\beta_{w-1,e+w} \end{bmatrix} \quad (8)$$

$$H = \begin{bmatrix} 1 & x_z & x_z^2 & \cdots & x_z^{e+w-1} \\ 1 & x_{z+1} & x_{z+1}^2 & \cdots & x_{z+1}^{e+w-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{z+e-1} & x_{z+e-1}^2 & \cdots & x_{z+e-1}^{e+w-1} \\ 0 & 1 & 2x_0 & \cdots & (e+w-1)x_0^{e+w-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{w-1} & \cdots & (e+w-1)x_{w-1}^{e+w-2} \end{bmatrix} \quad (9)$$

From the matrix Equation (7), the columns of $E=H^{-1}$ give the continuous coefficients of the continuous scheme of Equation (3).

DEVELOPMENT OF EBAMM INCORPORATING ONE EXTENDED FUTURE POINT FOR *K*=2

Here, we incorporate one extended future point at $x=x_z+3$ as a collocation point, thus the interpolation point, $e=1$ and the collocation points $w=4$ are considered. Therefore, Equation (3) becomes

$y(x)=\alpha_1(x)y_{z+1}+b[\beta_0(x)_{fz}+\beta_1(x)_{fz+1}+\beta_2(x)_{fz+2}+y_3(x)g_{z+3}$ (10)

The matrix H in Equation (9) becomes

$$H = \begin{pmatrix} 1 & x_z+b & (x_z+b)^2 & (x_z+b)^3 & (x_z+b)^4 \\ 0 & 1 & 2x_z & 3x_z^2 & 4x_z^3 \\ 0 & 1 & 2x_z+2b & 3(x_z+b)^3 & 4(x_z+b)^3 \\ 0 & 1 & 2x_z+4b & 3(x_z+2b)^2 & 4(x_z+2b)^3 \\ 0 & 1 & 2x_z+6b & 3(x_z+3b)^2 & 4(x_z+3b)^3 \end{pmatrix} \quad (11)$$

The inverse of the matrix $E=H^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using Equation (3), evaluating and simplifying it at $x=x_z$, $x=x_{z+2}$ and $x=x_{z+3}$, the following discrete schemes are obtained

$$y_z = y_{z+1} - \frac{3}{8}bf_z - \frac{19}{24}bf_{z+1} + \frac{5}{24}bf_{z+2} - \frac{1}{24}bf_{z+3}$$

$$y_{z+2} = y_{z+1} - \frac{1}{24}bf_z + \frac{13}{24}bf_{z+1} + \frac{13}{24}bf_{z+2} - \frac{1}{24}bf_{z+3}$$

$$y_{z+3} = y_{z+1} + \frac{1}{3}bf_{z+1} + \frac{4}{3}bf_{z+2} + \frac{1}{3}bf_{z+3} \qquad (12)$$

DEVELOPMENT OF EBAMM INCORPORATING ONE EXTENDED FUTURE POINT FOR $K = 3$

Here, we incorporate one extrapolated future point at $x=x_{z+4}$ as a collocation point, thus the interpolation point, e=1 and the collocation points w=5 are considered. Therefore, Equation (3) becomes

$$y(x)=\alpha_2(x)y_{z+2}+d[\beta_0(x)_{fz}+\beta_1(x)_{fz+1}+\beta_2(x)_{fz+2}+\beta_3(x)_{fz+3}+y_4(x)g_{z+4} \qquad (13)$$

The matrix H in Equation (9) becomes

$$H=\begin{pmatrix} 1 & x_z+2b & (x_z+2b)^2 & (x_z+2b)^3 & (x_z+2b)^4 & (x_z+2b)^5 \\ 0 & 1 & 2x_z & 3x_z^2 & 4x_z^3 & 5x_z^4 \\ 0 & 1 & 2x_z+2b & 3(x_z+b)^2 & 4(x_z+b)^3 & 5(x_z+b)^4 \\ 0 & 1 & 2x_z+4b & 3(x_z+2b)^2 & 4(x_z+2b)^3 & 5(x_z+2b)^4 \\ 0 & 1 & 2x_z+6b & 3(x_z+3b)^2 & 4(x_z+3b)^3 & 5(x_z+3b)^4 \\ 0 & 1 & 2x_z+8b & 3(x_z+4b)^2 & 4(x_z+4b)^3 & 5(x_z+4b)^4 \end{pmatrix} \qquad (14)$$

The inverse of the matrix $E=H^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using Equation (3), evaluating and simplifying it at $x=x_z$, $x=x_{z+3}$ and $x=x_{z+4}$, the following discrete schemes are obtained

$$y_z = y_{z+2} - \frac{29}{90}bf_z - \frac{62}{45}bf_{z+1} - \frac{4}{15}bf_{z+2} - \frac{2}{45}bf_{z+3} + \frac{1}{90}bf_{z+4}$$

$$y_{z+1} = y_{z+2} + \frac{19}{720}bf_z - \frac{173}{360}bf_{z+1} - \frac{19}{30}bf_{z+2} + \frac{37}{360}bf_{z+3} - \frac{11}{720}bf_{z+4}$$

$$y_{z+3} = y_{z+2} + \frac{11}{720}bf_z - \frac{37}{360}bf_{z+1} + \frac{19}{30}bf_{z+2} + \frac{173}{360}bf_{z+3} - \frac{19}{720}bf_{z+4}$$

$$y_{z+4} = y_{z+2} - \frac{1}{90}bf_z + \frac{2}{45}bf_{z+1} + \frac{4}{15}bf_{z+2} + \frac{62}{45}bf_{z+3} + \frac{29}{90}bf_{z+4} \qquad (15)$$

DEVELOPMENT OF EBAMM INCORPORATING ONE EXTENDED FUTURE POINT FOR $K = 4$

Here, we incorporate one extrapolated future point at $x=x_{z+5}$ as a collocation point, thus the interpolation point, e=1 and the collocation points w=6 are considered. Therefore, Equation (3) becomes

$$y(x)=\alpha_3(x)y_{z+3}+b[\beta_0(x)_{fz}+\beta_1(x)_{fz+1}+\beta_2(x)_{fz+2}+\beta_3(x)_{fz+3}+\beta_4(x)_{fz+4}+y_5(x)g_{z+5} \qquad (16)$$

The matrix H in Equation (9) becomes

$$H=\begin{pmatrix} 1 & x_z+3b & (x_z+3b)^2 & (x_z+3b)^3 & (x_z+3b)^4 & (x_z+3b)^5 & (x_z+3b)^6 \\ 0 & 1 & 2x_z & 3x_z^2 & 4x_z^3 & 5x_z^4 & 6x_z^5 \\ 0 & 1 & 2x_z+2b & 3(x_z+b)^2 & 4(x_z+b)^3 & 5(x_z+b)^4 & 6(x_z+b)^5 \\ 0 & 1 & 2x_z+4b & 3(x_z+2b)^2 & 4(x_z+2b)^3 & 5(x_z+2b)^4 & 6(x_z+2b)^5 \\ 0 & 1 & 2x_z+6b & 3(x_z+3b)^2 & 4(x_z+3b)^3 & 5(x_z+3b)^4 & 6(x_z+3b)^5 \\ 0 & 1 & 2x_z+8b & 3(x_z+4b)^2 & 4(x_z+4b)^3 & 5(x_z+4b)^4 & 6(x_z+4b)^5 \\ 0 & 1 & 2x_z+10b & 3(x_z+5b)^2 & 4(x_z+5b)^3 & 5(x_z+5b)^4 & 6(x_z+5b)^5 \end{pmatrix} \qquad (17)$$

The inverse of the matrix $E=H^{-1}$ is examined using Maple 18 from which the continuous scheme is obtained using Equation (3), evaluating and simplifying it at $x=x_z$, $x=x_{z+1}$, x_{z+2}, x_{z+4} and $x=x_{z+5}$, the following discrete schemes are obtained

$$y_z = y_{z+3} - \frac{51}{160}bf_z - \frac{219}{160}bf_{z+1} - \frac{57}{80}bf_{z+2} - \frac{57}{80}bf_{z+3} + \frac{21}{160}bf_{z+4} - \frac{3}{160}bf_{z+5}$$

$$y_{z+1} = y_{z+3} + \frac{1}{90}bf_z - \frac{17}{45}bf_{z+1} - \frac{19}{15}bf_{z+2} - \frac{17}{45}bf_{z+3} + \frac{1}{90}bf_{z+4}$$

$$y_{z+2} = y_{z+3} - \frac{11}{1440}bf_z + \frac{31}{480}bf_{z+1} - \frac{401}{720}bf_{z+2} - \frac{401}{720}bf_{z+3} + \frac{31}{480}bf_{z+4} - \frac{11}{1440}bf_{z+5}$$

$$y_{z+4} = y_{z+3} - \frac{11}{1440}bf_z - \frac{77}{1440}bf_{z+1} - \frac{43}{240}bf_{z+2} + \frac{511}{720}bf_{z+3} + \frac{637}{1440}bf_{z+4} - \frac{3}{160}bf_{z+5}$$

$$y_{z+5} = y_{z+3} + \frac{1}{90}bf_z - \frac{1}{15}bf_{z+1} + \frac{7}{45}bf_{z+2} + \frac{7}{45}bf_{z+3} + \frac{43}{30}bf_{z+4} + \frac{14}{45}bf_{z+5} \quad (18)$$

CONVERGENCE ANALYSIS

Here, the investigations of order, error constant, consistency, zero stability, and region of the absolute stability of Equations (12), (15), and (18) are worked-out.

ORDER AND ERROR CONSTANT

The order and error constants for Equation (12) are obtained as follows

$C_0 = C_1 = C_2 = C_3 = C_4 = (000)^T$ but

$$c_5 = \begin{pmatrix} \frac{19}{720} & \frac{11}{720} & -\frac{1}{90} \end{pmatrix}^T$$

Therefore, Equation (12) has order $p=4$ and error constants, $\frac{19}{720}$ $\frac{11}{720}$ $-\frac{1}{90}$

Applying the same approach to Equation (15), we obtained
$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = (0000)^T$ but

$$c_6 = \begin{pmatrix} -\frac{1}{90} & \frac{11}{1440} & \frac{11}{1440} & -\frac{1}{90} \end{pmatrix}^T$$

Therefore, Equation (15) has order $p=5$ and error constants, $-\frac{1}{90}$ $\frac{11}{1440}$ $\frac{11}{1440}$ $-\frac{1}{90}$

Applying the same approach to Equation (18), we obtained
$C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = (00000)^T$ but

$$c_7 = \begin{pmatrix} -\frac{3}{32} & 0 & -\frac{11}{288} & -\frac{3}{32} & -\frac{197}{18} \end{pmatrix}^T$$

Therefore, Equation (18) has order $p=6$ and error constants, $-\frac{3}{32}$ 0 $-\frac{11}{288}$ $-\frac{3}{32}$ $-\frac{197}{18}$

CONSISTENCY

Since the schemes in Equations (12), (15), and (18) satisfy the condition for consistency of order $a \geq 1$, then they are consistent.

Stability analysis

The zero stability for Equation (12) is evaluated as follows

$$\begin{pmatrix}-1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}\begin{pmatrix}y_{z+1}\\ y_{z+2}\\ y_{z+3}\end{pmatrix}=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}y_{z-2}\\ y_{z-1}\\ y_{z}\end{pmatrix}+$$

$$b\begin{pmatrix}-\frac{19}{24} & \frac{5}{24} & -\frac{1}{24}\\ \frac{13}{24} & \frac{13}{24} & -\frac{1}{24}\\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}f_{z+1}\\ f_{z+2}\\ f_{z+3}\end{pmatrix}+b\begin{pmatrix}0 & 0 & -\frac{3}{8}\\ 0 & 0 & -\frac{1}{24}\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}f_{z-2}\\ f_{z-1}\\ f_{z}\end{pmatrix}$$

where

$$Q_2^{(1)}=\begin{pmatrix}-1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}, Q_1^{(1)}=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$R_2^{(1)}=\begin{pmatrix}-\frac{19}{24} & \frac{5}{24} & -\frac{1}{24}\\ \frac{13}{24} & \frac{13}{24} & -\frac{1}{24}\\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3}\end{pmatrix}$$

$$\text{and } S_2^{(1)}=\begin{pmatrix}0 & 0 & -\frac{3}{8}\\ 0 & 0 & -\frac{1}{24}\\ 0 & 0 & 0\end{pmatrix}$$

The first characteristic polynomial is given by

$$\begin{aligned}G(v)&=\det\left(vQ_2^{(1)}-Q_1^{(1)}\right)\\ &=\left|vQ_2^{(1)}-Q_1^{(1)}\right|=0.\end{aligned}\tag{19}$$

Now, we have,

$$G(v)=\left|v\begin{pmatrix}-1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}-\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right|=$$

$$\left|\begin{pmatrix}-v & 0 & 0\\ -v & v & 0\\ -v & 0 & v\end{pmatrix}-\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right|$$

$$\Rightarrow G(v)=\begin{pmatrix}-v & 0 & -1\\ -v & v & 0\\ -v & 0 & v\end{pmatrix}$$

The following are obtained using Maple 18 software,

$$G(v)=-v^3-v^2\Rightarrow -v^3-v^2=0$$

$\Rightarrow v_1=-1, v_2=0, v_3=0$. Since $|v_i|<1$, i=1,2,3, (12) is zero stable.

By the same procedure for Equation (15)

$$\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & -1 & 0 & 1\end{pmatrix}\begin{pmatrix}y_{z+1}\\ y_{z+2}\\ y_{z+3}\\ y_{z+4}\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}y_{z-3}\\ y_{z-2}\\ y_{z-1}\\ y_{z}\end{pmatrix}$$

$$+b\begin{pmatrix}-\frac{62}{45} & -\frac{4}{15} & -\frac{2}{45} & \frac{1}{90}\\ -\frac{173}{360} & -\frac{19}{30} & \frac{37}{360} & -\frac{11}{720}\\ -\frac{37}{360} & \frac{19}{30} & \frac{173}{360} & -\frac{19}{720}\\ \frac{2}{45} & \frac{4}{15} & \frac{62}{45} & \frac{29}{90}\end{pmatrix}\begin{pmatrix}f_{z+1}\\ f_{z+2}\\ f_{z+3}\\ f_{z+4}\end{pmatrix}+$$

$$b\begin{pmatrix}0 & 0 & 0 & -\frac{29}{90}\\ 0 & 0 & 0 & \frac{19}{720}\\ 0 & 0 & 0 & \frac{11}{720}\\ 0 & 0 & 0 & -\frac{1}{90}\end{pmatrix}\begin{pmatrix}f_{z-3}\\ f_{z-2}\\ f_{z-1}\\ f_{z}\end{pmatrix}$$

where

$$Q_2^{(2)}=\begin{pmatrix}0&-1&0&0\\1&-1&0&0\\0&-1&1&0\\0&-1&0&1\end{pmatrix},Q_1^{(2)}=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$R_2^{(2)}=\begin{pmatrix}-\frac{62}{45}&-\frac{4}{15}&-\frac{2}{45}&\frac{1}{90}\\-\frac{173}{360}&-\frac{19}{30}&\frac{37}{360}&-\frac{11}{720}\\-\frac{37}{360}&\frac{19}{30}&\frac{173}{360}&-\frac{19}{720}\\\frac{2}{45}&\frac{4}{15}&\frac{62}{45}&\frac{29}{90}\end{pmatrix}$$

and

$$S_2^{(2)}=\begin{pmatrix}0&0&0&-\frac{29}{90}\\0&0&0&\frac{19}{720}\\0&0&0&\frac{11}{720}\\0&0&0&-\frac{1}{90}\end{pmatrix}$$

The first characteristic polynomial is presented as

$$\begin{aligned}G(v)&=\det\left(vQ_2^{(2)}-Q_1^{(2)}\right)\\&=\left|vQ_2^{(2)}-Q_1^{(2)}\right|=0.\end{aligned}\tag{20}$$

Now, we have,

$$G(v)=\left|v\begin{pmatrix}0&-1&0&0\\1&-1&0&0\\0&-1&1&0\\0&-1&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right|=$$

$$\left|\begin{pmatrix}0&-v&0&0\\v&-v&0&0\\0&-v&v&0\\0&-v&0&v\end{pmatrix}-\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\right|$$

$$\Rightarrow G(v)=\begin{pmatrix}0&-v&0&-1\\v&-v&0&0\\0&-v&v&0\\0&-v&0&v\end{pmatrix}$$

Using Maple 18 software, we obtain

$G(v)=v^4+v^3\Rightarrow v^4+v^3=0$

$\Rightarrow v_1=-1,v_2=0,v_3=0,v_4=0$. Since $|v_i|<1$, i=1,2,3,4, Equation (15) is zero stable.

Following the same procedure for Equation (18)

$$\begin{pmatrix}0&0&-1&0&0\\1&0&-1&0&0\\0&1&-1&0&0\\0&0&-1&1&0\\0&0&-1&0&1\end{pmatrix}\begin{pmatrix}y_{z+1}\\y_{z+2}\\y_{z+3}\\y_{z+4}\\y_{z+5}\end{pmatrix}=\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}y_{z-4}\\y_{z-3}\\y_{z-2}\\y_{z-1}\\y_z\end{pmatrix}$$

$$+b\begin{pmatrix}-\frac{62}{45}&-\frac{4}{15}&-\frac{2}{45}&\frac{1}{90}\\-\frac{173}{360}&-\frac{19}{30}&\frac{37}{360}&-\frac{11}{720}\\-\frac{37}{360}&\frac{19}{30}&\frac{173}{360}&-\frac{19}{720}\\\frac{2}{45}&\frac{4}{15}&\frac{62}{45}&\frac{29}{90}\end{pmatrix}\begin{pmatrix}f_{z+1}\\f_{z+2}\\f_{z+3}\\f_{z+4}\end{pmatrix}+$$

$$b\begin{pmatrix}0&0&0&-\frac{29}{90}\\0&0&0&\frac{19}{720}\\0&0&0&\frac{11}{720}\\0&0&0&-\frac{1}{90}\end{pmatrix}\begin{pmatrix}f_{z-3}\\f_{z-2}\\f_{z-1}\\f_z\end{pmatrix}$$

where

$$Q_2^{(2)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, Q_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_2^{(2)} = \begin{pmatrix} -\frac{62}{45} & -\frac{4}{15} & -\frac{2}{45} & \frac{1}{90} \\ -\frac{173}{360} & -\frac{19}{30} & \frac{37}{360} & -\frac{11}{720} \\ -\frac{37}{360} & \frac{19}{30} & \frac{173}{360} & -\frac{19}{720} \\ \frac{2}{45} & \frac{4}{15} & \frac{62}{45} & \frac{29}{90} \end{pmatrix}$$

and

$$S_2^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{51}{160} \\ 0 & 0 & 0 & 0 & \frac{1}{90} \\ 0 & 0 & 0 & 0 & -\frac{11}{1440} \\ 0 & 0 & 0 & 0 & -\frac{11}{1440} \\ 0 & 0 & 0 & 0 & \frac{1}{90} \end{pmatrix}$$

The first characteristic polynomial is presented as

$$\begin{aligned} G(v) &= \det\left(vQ_2^{(3)} - Q_1^{(3)}\right) \\ &= \left|vQ_2^{(3)} - Q_1^{(3)}\right| = 0. \end{aligned} \tag{21}$$

Now, we have,

$$G(v) = \left| v\begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right|$$

$$= \left\| \begin{pmatrix} 0 & 0 & -v & 0 & 0 \\ v & 0 & -v & 0 & 0 \\ 0 & v & -v & 0 & 0 \\ 0 & 0 & -v & v & 0 \\ 0 & 0 & -v & 0 & v \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\|$$

$$\Rightarrow G(v) = \begin{pmatrix} 0 & 0 & -v & 0 & -1 \\ v & 0 & -v & 0 & 0 \\ 0 & v & -v & 0 & 0 \\ 0 & 0 & -v & v & 0 \\ 0 & 0 & -v & 0 & v \end{pmatrix}$$

Using Maple 18 software, we obtain

$$G(v) = -v^5 - v^4 \Rightarrow -v^5 - v^4 = 0$$

$\Rightarrow v_1 = -1, v_2 = 0, v_3 = 0, v_4 = 0, v_5 = 0$. Since $|v_i|<1$, i=1,2,3,4,5, Equation (18) is zero stable.

CONVERGENCE

Since Equations (12), (15), and (18) are both consistent and zero stable, therefore, they are convergent.

REGION OF ABSOLUTE STABILITY

The regions of absolute stability of the numerical methods for DDEs are considered. We considered finding the P- and Q-stability by applying Equations (12), (15), and (18) to the DDEs of this form

$$\begin{aligned} y'(t) &= uy(\mathrm{t}) + \mathrm{c}y(t-\tau), & t \geq t_0 \\ y(t) &= m(t), & t \leq t_0 \end{aligned} \tag{22}$$

Where, $m(t)$ is the initial function, w, z are complex coefficients, $\tau = zb$, $z \in \mathbb{Z}^+$, b is the step size, and $z = \frac{\tau}{b}, z$ is a positive integer. Let $P_1 = bu$ and $P_2 = bc$, then

Making use of Maple 18 and MATLAB, the region of P- and Q-stability for Equations (12), (15), and (18) are plotted and shown in Figures 1-6.

The P-stability regions in Figures 1-3 lie inside the open-ended region while the Q-stability regions in Figures 4-6 lie inside the enclosed region as shown below.

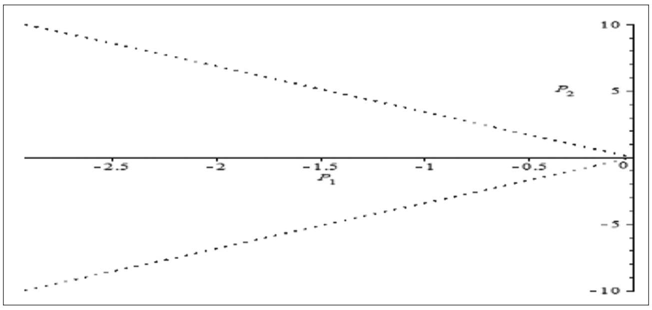

Figure 1: Region of P-stability (EBAMM) in Equation (12)

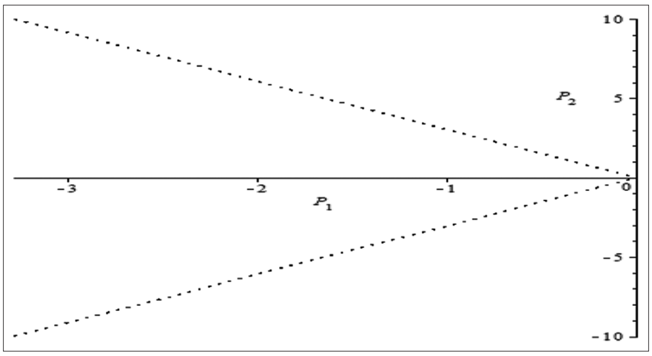

Figure 2: Region of P-stability (EBAMM) in Equation (15)

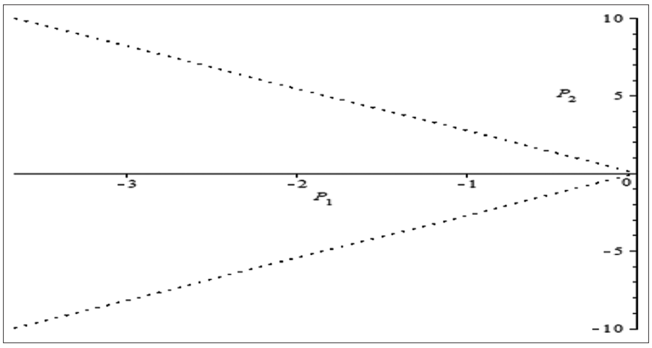

Figure 3: Region of P-stability (EBAMM) in Equation (18)

IMPLEMENTATION OF NUMERICAL PROBLEMS

In this section, some first-order DDEs shall be solved using Equations (12), (15), and (18) of the discrete schemes that have been established. The delay argument shall be evaluated using the idea of sequence formulated by Sirisena and Yakubu.[8]

Problem 1

$$y'(t) = -1000y(t) + y(t - (\ln(1000 - 1))), \ 0 \le t \le 3$$

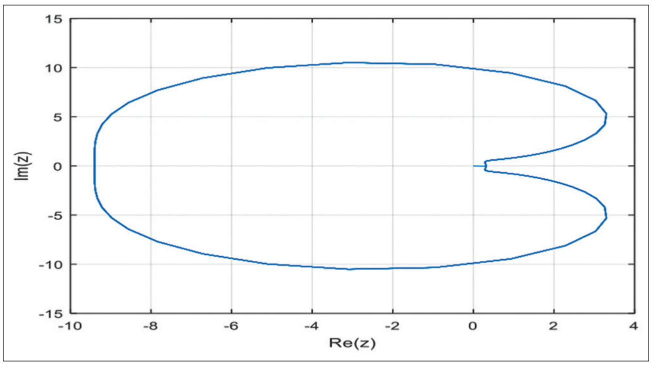

Figure 4: Region of Q-stability (EBAMM) in Equation (12)

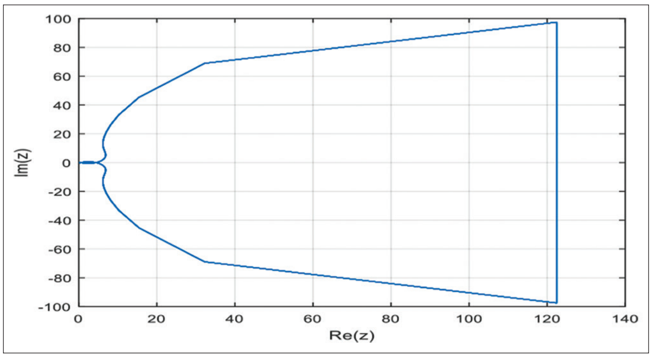

Figure 5: Region of Q-stability (EBAMM) in Equation (15)

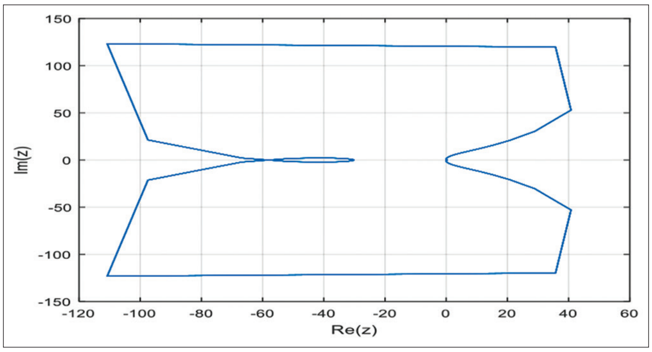

Figure 6: Region of Q-stability (EBAMM) in Equation (18)

$y(t)=e^{-t}$, $t\leq0$

Exact solution $y(t)=e^{-t}$ in[8]

Problem 2

$$y'(t)=-1000y(t)+997e^{-3}y(t-1)+(1000-997e^{-3}),$$

$0\leq t\leq3$

$y(t)=1+e^{-3t}$, $t\leq0$

Exact solution $y(t)=1+e^{-3t}$ in[8]

ANALYSIS AND COMPARISON OF RESULTS

Here, the solutions of the schemes derived in Equations (12), (15), and (18), shall be investigated in solving the two problems above by estimating their absolute errors.

The analysis of results is obtained by determining absolute differences of the exact solutions and the numerical solutions.[14] The results obtained after the application of the proposed method shall be compared to other existing methods studied by Osu *et al*., Chibuisi *et al*., Onumanyi *et al*.[9,10,13] to prove its superiority. The notations used in the table are stated below

EABMM = Extrapolated Block Adams Moulton Methods for step numbers k=2,3, and 4.

RBBDFM = Reformulated Block Backward Differentiation Formulae Methods for step numbers k=3 and 4 in Sirisena and Yakubu.[8]

TDBBDFM = Third derivative block backward differentiation formulae method for step numbers k=2,3, and 4 in Osu *et al*.[9]

CBBDFM = Conventional Block Backward Differentiation Formulae Method for step numbers k=2 and 3 in Onumanyi *et al*.[13]

MAXE = Maximum Error.

Table 1: Comparison between the maximum absolute errors of EBAMM k=2,3, and 4 with Sirisena and Yakubu, Osu *et al*., Onumanyi *et al*.[8,9,13] for constant step size d=0.01 using Problem 1

Computational method	Compared maxes with Sirisena and Yakubu, Osu *et al*., Onumanyi *et al*. [8,9,13]
EBAMM MAXE for k=2	7.69E-10
EBAMM MAXE for k=3	8.23E-11
EBAMM MAXE for k=4	6.45E-13
RBBDFMAXE for k=3	4.88E-06
RBBDF MAXE for k=4	4.38E-06
TDBBDFM MAXE for k=2	3.44E-03
TDBBDFM MAXE for k=3	6.32E-03
TDBBDFM MAXE for k=4	9.64E-03
CBBDF MAXE for k=2	8.96E-05
CBBDF MAXE for k=3	9.39E-06

CPU time of EAMM for k=2 is 0.312s, k=3 is 0.285s, and k=4 is 0.208s

Table 2: Comparison between the maximum absolute errors of EBAMM k=2,3, and 4 with Sirisena and Yakubu, Osu *et al*., Onumanyi *et al*.[8,9,13] for constant step size d=0.01 using Problem 2

Computational method	Compared maxes with Sirisena and Yakubu, Osu *et al*., Onumanyi *et al*.[8,9,13]
EBAMM MAXE for k=2	7.69E-11
EBAMM MAXE for k=3	8.23E-13
EBAMM MAXE for k=4	6.45E-14
RBBDFMAXE for k=3	1.54E-09
RBBDF MAXE for k=4	1.04E-09
TDBBDFM MAXE for k=2	3.44E-03
TDBBDFM MAXE for k=3	6.29E-03
TDBBDFM MAXE for k=4	9.64E-03
CBBDF MAXE for k=2	6.32E-06
CBBDF MAXE for k=3	5.10E-07

CPU time of EAMM for k=2 is 0.310s, k=3 is 0.290s, and k=4 is 0.212s

CONCLUSION

The discrete schemes of Equations (12), (15), and (18) were deduced from their different continuous formulations and were examined to be convergent, *P*- and *Q*-stable. Furthermore, it was observed in Tables 1 and 2 that the EBAMM for k=4 scheme performed better than the EBAMM schemes for step numbers k=3 and k=2 when compared with other existing methods. It is recommended that the EBAMM schemes of higher step numbers perform better than the EBAMM schemes of lower step numbers and also the step numbers of k=2,3 and k=4 are suitable for solving DDEs. Further studies should be carried out for step numbers k=5,6,7,… on the construction of discrete schemes of EBAMM for numerical solutions of DDEs without the introduction of interpolation techniques in evaluating the delay term.

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