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RESEARCH ARTICLE

Riemann Integrals of Powers of Metallic Ratios

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ABSTRACT

Metallic ratios are important class of numbers in mathematics which were discussed extensively right from ancient times. In this paper, treating the expression for *k*th metallic ratio as a continuous function, I had found new results for determining Riemann integrals of such functions over compact intervals of the form [0, *n*] for any natural number *n*.

Key words: Metallic Ratios, Recurrence Relation, Riemann Integrals, Compact Interval, Integration by Parts

INTRODUCTION

The study of golden ratio has been a subject from ancient time to modern times and had its influence in almost every branch of engineering and science. The sequence of numbers called metallic ratios in which golden ratio is its first term has created so much interest among professionals as well as amateur mathematicians and has led to various useful applications. Several papers in recent decades have been published related to the study of metallic ratios. In this paper, I will prove certain new results regarding determining Riemann integral of powers of metallic ratios.

DEFINITION

2.1 For any natural number k, the kth metallic ratio is a number defined as the positive root of the equation $x^2-kx-1=0$ (2.1). If we denote the kth metallic ratio by M_K then from Equation (2.1), we obtain

$$M_k = \frac{k+\sqrt{k^2+4}}{2} \quad (2.2),$$

$$-\frac{1}{M_k} = \frac{k-\sqrt{k^2+4}}{2} \quad (2.3).$$

2.2 For the values of $k = 1, 2, 3$ in Equation (2.2), the numbers obtained are called as golden, silver, and bronze ratios, respectively. Thus, the numbers $M_1 = \frac{1+\sqrt{5}}{2}, M_2 = 1+\sqrt{2}, M_3 = \frac{3+\sqrt{13}}{2}$ are golden, silver, and bronze ratios, respectively. In this sense, golden, silver, and bronze ratios are special class of sequence of metallic ratios whose terms are given by Equation (2.2). For knowing more about metallic ratios and their properties.[1-7]

RIEMANN INTEGRAL OF FIRST POWER OF METALLIC RATIOS

3.1 Theorem 1

If n is any natural number and if M_K is the kth metallic number, then

$$\int_{k=0}^{n} M_k \, dk = \frac{nM_n}{2} + \log_e(M_n) \quad (3.1)$$

Proof: Using Equation (2.2), integrating M_K over [0, n] we get

$$\int_{k=0}^{n} M_k \, dk = \int_{k=0}^{n} \left(\frac{k+\sqrt{k^2+4}}{2}\right) dk = \frac{n^2}{4} + \frac{1}{2}\left[\frac{k\sqrt{k^2+4}}{2} + 2\log\left(k+\sqrt{k^2+4}\right)\right]_{k=0}^{n}$$

$$= \frac{n}{2}\left(\frac{n+\sqrt{n^2+4}}{2}\right) + \log_e\left(\frac{n+\sqrt{n^2+4}}{2}\right)$$

$$= \frac{nM_n}{2} + \log_e(M_n)$$

This completes the proof.

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3.2 Theorem 2

If n is any natural number and if M_K is the kth metallic number, then

$$\int_{k=0}^{n} \frac{1}{M_k}\,dk = \frac{n}{2M_n} + \log_e\left(M_n\right) \quad (3.2)$$

Proof: Using Equation (2.3), integrating $\frac{1}{M_k}$ over [0, n] we get

$$\int_{k=0}^{n} \frac{1}{M_k}\,dk = \int_{k=0}^{n} \left(\frac{\sqrt{k^2+4}-k}{2}\right)dk$$

$$= \frac{1}{2}\left[\frac{k\sqrt{k^2+4}}{2} + 2\log\left(k+\sqrt{k^2+4}\right)\right]_{k=0}^{n} - \frac{n^2}{4}$$

$$= \frac{n}{2}\left(\frac{\sqrt{n^2+4}-n}{2}\right) + \log_e\left(\frac{n+\sqrt{n^2+4}}{2}\right)$$

$$= \frac{n}{2M_n} + \log_e\left(M_n\right)$$

This completes the proof.

RIEMANN INTEGRAL OF SECOND POWER OF METALLIC RATIOS

4.1 Theorem 3

If n is any natural number and if M_k is the kth metallic number, then

$$\int_{k=0}^{n} M_k^2\,dk = \frac{\left(n^3+6n-8\right)+\left(2M_n-n\right)^3}{6} \quad (4.1)$$

Proof: Using Equation (2.2), integrating M_K over [0, n] we get

$$\int_{k=0}^{n} M_k^2\ dk = \int_{k=0}^{n} \left(\frac{k+\sqrt{k^2+4}}{2}\right)^2 dk = \frac{1}{4}$$

$$\int_{k=0}^{n} \left(2k^2+4+2k\sqrt{k^2+4}\right)dk = \frac{n^3}{6} + n + \frac{1}{2}$$

$$\int_{k=0}^{n} \sqrt{k^2+4}\times k dk = \frac{n^3}{6} + n + \frac{1}{2}\int_{u=2}^{\sqrt{n^2+4}} (u)(u\,du) = \frac{n^3}{6}$$

$$+n+\frac{1}{6}\left(\left(n^2+4\right)^{3/2}-8\right) = \frac{(n^3+6n-8)+\left(n^2+4\right)^{3/2}}{6}$$

$$= \frac{(n^3+6n-8)+\left(2M_n-n\right)^3}{6}$$

This completes the proof.

4.2 Theorem 4

If n is any natural number and if M_K is the kth metallic number, then

$$\int_{k=0}^{n} \frac{1}{M_k^2}\,dk = \frac{\left(n^3+6n+8\right)-\left(2M_n-n\right)^3}{6} \quad (4.2)$$

Proof: Using Equation (2.3), integrating $\frac{1}{M_k^2}$ over [0, n] we get

$$\int_{k=0}^{n} \frac{1}{M_k^2}\ dk = \int_{k=0}^{n} \left(\frac{k-\sqrt{k^2+4}}{2}\right)^2 dk$$

$$= \frac{1}{4}\int_{k=0}^{n} \left(2k^2+4-2k\sqrt{k^2+4}\right)dk$$

$$= \frac{n^3}{6} + n - \frac{1}{2}\int_{k=0}^{n} \sqrt{k^2+4}\times k dk$$

$$= \frac{n^3}{6} + n - \frac{1}{2}\int_{u=2}^{\sqrt{n^2+4}} (u)(u\,du)$$

$$= \frac{n^3}{6} + n - \frac{1}{6}\left(\left(n^2+4\right)^{3/2}-8\right)$$

$$= \frac{(n^3+6n+8)-\left(n^2+4\right)^{3/2}}{6}$$

$$= \frac{(n^3+6n+8)-\left(2M_n-n\right)^3}{6}$$

This completes the proof.

RIEMANN INTEGRAL OF THIRD POWER OF METALLIC RATIOS

5.1 Theorem 5

If n is any natural number and if M_K is the kth metallic number, then

$$\int_{k=0}^{n} M_k^3\,dk = \frac{n\left(2M_n-n\right)^3+\left(n^4+6n^2\right)}{8} \quad (5.1)$$

Proof: Using Equation (2.2), integrating M_k over [0, n] we get

$$\int_{k=0}^{n} M_k^3\,dk = \int_{k=0}^{n}\left(\frac{k+\sqrt{k^2+4}}{2}\right)^3 dk$$

$$= \frac{1}{8}\int_{k=0}^{n}\left(k^3+3k^2\sqrt{k^2+4}+3k(k^2+4)+\left(k^2+4\right)^{3/2}\right)$$

$$dk = \frac{1}{8}\int_{k=0}^{n}\left(4k^3+12k\right)dk+\frac{3}{8}\int_{k=0}^{n}k^2\sqrt{k^2+4}\,dk+$$

$$\frac{1}{8}\int_{k=0}^{n}\left(k^2+4\right)^{3/2}dk = \frac{n^4+6n^2}{8}$$

$$+\frac{1}{2}\int_{k=0}^{n}\left(k^2+4\right)^{3/2}dk-\frac{3}{2}\int_{k=0}^{n}\sqrt{k^2+4}\,dk$$

I now try to calculate the second term $\int_{k=0}^{n}\left(k^2+4\right)^{3/2}dk$ using integration by parts formula

$$\int_{k=0}^{n}\left(k^2+4\right)^{3/2}dk = \left[k\left(k^2+4\right)^{3/2}\right]_{k=0}^{n}$$

$$-\int_{k=0}^{n}3k^2\sqrt{k^2+4}\,dk = n\left(n^2+4\right)^{3/2}$$

$$-3\int_{k=0}^{n}\left(k^2+4\right)^{3/2}dk+12\int_{k=0}^{n}\sqrt{k^2+4}\,dk$$

Hence, we get

$$\int_{k=0}^{n}\left(k^2+4\right)^{3/2}dk = \frac{n\left(n^2+4\right)^{3/2}}{4}+3\int_{k=0}^{n}\sqrt{k^2+4}\,dk$$

Substituting this value in the required expression we get

$$\int_{k=0}^{n} M_k^3\,dk = \frac{n^4+6n^2}{8}+\frac{n\left(n^2+4\right)^{3/2}}{8}$$

$$= \frac{\left(n^4+6n^2\right)+n\left(2M_n-n\right)^3}{8}$$

This completes the proof.

5.2 Theorem 6

If n is any natural number and if M_K is the kth metallic number, then

$$\int_{k=0}^{n}\frac{1}{M_k^3}\,dk = \frac{n\left(2M_n-n\right)^3-\left(n^4+6n^2\right)}{8} \quad (5.2)$$

Proof: Using (2.3), integrating $\frac{1}{M_k^3}$ over [0, n] we get

$$\int_{k=0}^{n}\frac{1}{M_k^3}\,dk = \int_{k=0}^{n}\left(\frac{\sqrt{k^2+4}-k}{2}\right)^3 dk$$

$$= \frac{1}{8}\int_{k=0}^{n}\begin{pmatrix}\left(k^2+4\right)^{3/2}-3k(k^2+4)\\ +3k^2\sqrt{k^2+4}-k^3\end{pmatrix}dk$$

$$= -\frac{1}{8}\int_{k=0}^{n}\left(4k^3+12k\right)dk+\frac{3}{8}\int_{k=0}^{n}k^2\sqrt{k^2+4}\,dk+$$

$$\frac{1}{8}\int_{k=0}^{n}\left(k^2+4\right)^{3/2}dk = -\left(\frac{n^4+6n^2}{8}\right)$$

$$+\frac{1}{2}\int_{k=0}^{n}\left(k^2+4\right)^{3/2}dk-\frac{3}{2}\int_{k=0}^{n}\sqrt{k^2+4}\,dk$$

Using the expression

$$\int_{k=0}^{n}\left(k^2+4\right)^{3/2}dk = \frac{n\left(n^2+4\right)^{3/2}}{4}+3\int_{k=0}^{n}\sqrt{k^2+4}\,dk$$

derived above we get

$$\int_{k=0}^{n}\frac{1}{M_k^3}\,dk = -\left(\frac{n^4+6n^2}{8}\right)+\frac{n\left(n^2+4\right)^{3/2}}{8}$$

$$= \frac{n\left(2M_n-n\right)^3-\left(n^4+6n^2\right)}{8}$$

This completes the proof.

CONCLUSION

In this paper, through theorems 1 to 6 established in sections 3, 4, and 5, I had obtained nice expressions for computing Riemann integral of the first three powers of metallic ratios and their corresponding reciprocals. These results are valid because the formula for kth metallic ratio in Equation (2.2) is a continuous function and any continuous function is Riemann integrable over a compact interval which in this case is [0, n]. One can try to generalize the formulas obtained for general integral powers of metallic ratios and their corresponding reciprocals. These results may provide new insights into already known properties of metallic ratios.

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