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The Critics and Contributions of Mathematical Philosophy in Hong Kong Secondary Education

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ABSTRACT

There are various schools of mathematical philosophy. However, none of them can be founded on mathematics alone. At the same time, there are two types of mathematical proof styles: Dialectic and algorithm mathematical proof. The relationship between proof and philosophy is to study philosophical problems with mathematical models. This type of proof is important to Hong Kong Secondary education. In addition, teachers should explain the connection between mathematics-based subjects, such as physics, so that lessons are more interesting rather than technical. Mathematics relates to nearly all other subjects, and as such has the role of a "public servant" when it comes to serving them. One role of mathematics is to act as a "rational" instrument for various subjects. This can be shown in many ancient human activities, such as Daoism and Hiu, together with their symbolic representations. These examples are similar to Jewish culture; when discussing confidence, Abraham is often mentioned due to being the "Father of Confidence." Thus, it may be said that mathematics is more than just a servant — it is also a cultural subject that has been recorded throughout history. At the same time, in ancient China, what Daoism tried to do was a searching for the unity between human and nature. This is a kind of Taiwan philosophy. It tells us that there are always connection human and nature. Thus, man should follow strictly to the rule of nature so that one can finally achieve the harmony between man and nature. To conclude, other than mathematical proof, Hong Kong teachers should also allow students to learn the cultural context behind various topics and subjects.

Key words: Algorithm mathematical proof, dialectic, mathematical philosophy

INTRODUCTION

Much has been said and written about changes in the teaching and learning of mathematics as well as its relation to science or more specifically, physics. However, those reasons will allow us to discuss the philosophy of mathematics. According to Webster's Dictionary (2003), philosophy is defined as: "The critical study of the basic principles and concepts of a particular branch of knowledge is, especially with a view to improving or reconstituting them" (p. 1455). Indeed, from examining philosophy in the present study, we can have a better understanding of those basic principles and concepts that a teacher should hold besides the field of mathematics (Fredette, 2009). According to Schoenfeld, 2004, there is controversy surrounding the reform of mathematics education. Davis and Mitchell suggested in 2008 that the controversy could be rooted in philosophical consideration. On the contrary, the occasion of explicitly studying philosophy is not just the philosophy of mathematics: "Is it possible that teachers' conceptions of mathematics need to undergo significant revisions before the teaching of mathematics can be revised?" (Davis and Mitchell, p. 146).

PHILOSOPHY OF MATHEMATICS

In 2004, Ernest proposed educators to think more profoundly about mathematics. He asked five basic questions about how mathematics is being taught: What is mathematics? How does mathematics relate to society? What is learning mathematics?

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What is teaching mathematics? What is the status of mathematics education as a field of knowledge? These challenging questions reflect on educators to not only transform their instructional practices but also to alter their own beliefs in mathematics and mathematical teaching (Fredette, 2009). Can the above questions be a catalyst for true reforms in mathematics education? To answer these questions, one must examine the philosophy of mathematics in depth.

Logicism

In terms of mathematics, logicism tries to reduce the subject into logic (Johansen, 2010). The work of logicist Gottlob Frege can, form the basis of this idea. However, there is a fatal blow about Frege's program-me, which is discovered by Bertrand Russell's paradox. Afterwards, Russell and Alfred North Whitehead carried out the logicist program and presented their concepts in the Principia Mathematical with three-volumes of work issued between 1910 and 1913. Indeed, logicism mainly suggests that all mathematical ideas can be reformulated. It means those ideas such as "number," "addition" and so forth can be expressed in logical (set-theoretical) terms. Hence, from these terms we can derive all mathematical theorems from several axioms using logical deduction. In other words, Russell and Whitehead attempt to view mathematics from a purely analytic a priori perspective. They divide the paradoxes of set theory by type theory where different types of sets are introduced as the following:

Type 1 sets: All elements are individuals

Type 2 sets: Elements are individuals or sets of type 1

Type 3 sets: Elements are individuals or sets of type 1 or type 2.

These result in the basic idea that there will be no sets containing sets of its own type as elements but only sets of the lower types (and individuals). Since there is no self-membership, Russell or Bureli Forti paradox will disappear.

To carry out the logicist program, Russell and Whitehead have introduced two non- tautological axioms which are the following:

The Axiom of Infinity: There is at least one actual infinite set. Since one lives in a finite world, with common sense in mind, it is not plausible.

The Axiom of Reducibility: This is introduced so that one can fix a problem caused by the adoption of type theory. One uses sets to contain numbers. Because of type theory, one will have different types of sets, and even the same numbers will be of different types. To cite an example, one may consider the number "3" as both type 2 number and type 3 number and so on. Therefore, Russell and Whitehead try to utilize an axiom stating that every set of a higher type is coextensive with sets of the lower levels – the Axiom of Reducibility. This means that the axiom is an ad hoc move for fixing a specific problem in the developed theory and is not a logical tautology. In such a case, one can solve the cumbersome problem of mathematics from a "modified and improved logical view" (Johansen, 2010).

From the above discussion, one may conclude that mathematics cannot be reduced to just simple logic or one does not recognize how to do it now.

Intuitionism

The basis of intuitionism is to provide a bottom-up interpretation of mathematics (Johansen, 2010). Initially, a Dutch mathematician named Luitzen Egbertus Jan Brouwer (1881–1966) developed the theory and began his doctoral dissertation from 1907. While his students Arend Heyting (1898–1980), and Herman Weyl (1885–1955) who were "converted" to intuitionism around 1920 also worked for it. From an idea formed by Leopold Kronecker (1823–1891) who holds a constructivist stance, Brouwer tries to re-establish mathematics from the base and upward. According to Brouwer, independent and real mathematical objects do not exist. Indeed, mathematics is a mental activity, and its objects are a mentally construction under the basic root of a Kantian intuition of time. In general, the intuitionist attempts to reconstruct mathematics from the following path:

1. One constructs natural numbers according to the basic intuition of time
2. Next, one constructs those rational numbers from naturals
3. Furthermore, one constructs the real number system
4. Finally, one constructs geometry from the real's using analytic geometry (Heyting 1931, pp.52; Shapiro 2000. p.177).

From the above, one can observe that those mathematical theories developed by the intuitionists were somehow unfamiliar to many of Brouwer's contemporaries. This means that theorems such as fundamental theorems, which are intuitively correct to most, may not hold true to those intuitionists. For example, theorems such as "Every number is smaller than, equal to or greater than zero" and "Every continuous curve defined on a closed interval has a maximum" were disproved by Brouwer in 1923, pp.337.

Brouwer suggested that one should have an adjustment for the allowed basic logical inference. According to Brouwer, one could only apply the principle *tertium non datur* in finite cases, but not infinite cases (ibid. p. 336). This means that the strong tool of oblique evidence in the infinite case elaborated by intuitionists was scrambled.

Nevertheless, intuitionism does not only focus on ordinary mathematics, it also proposes to work for theorems that seem clearly wrong or even extraordinary from a long-established perspective. From the intuitionists' perspective, every real function is continuous (Feferman, 1998. p.47).

Certainly, there is much discussion on whether one can consider private intuitions as a safe foundation for mathematics. The main criticisms arise from the results that intuitionism produces or the results it does not produce. As a result, the constructive features of intuitionism appear to place serious and unnecessary limitations on mathematics.

From the above discussions, intuitionism fails as a theory. Not only because of its invalidity to supply safe foundations for mathematics but also due to its failure of producing foundations for the body of knowledge which has been already known as mathematical truth.

Formalism

Formalism can be understood as another fundamental thought of mathematical philosophy. This was initially denoted by the Hilbert-program who elaborated it on David Hilbert's announcement on the Axionatisches Denken in 1917 (Hilbert, 1918). In the years following, Hilbert also published several other announcements. A number of mathematicians such as John von Neumann (1903–1957) and Paul Bernays (1888–1977) took up Hilbert's program during 1920's.

The program was motivated by the discovery of the paradoxes in set theory.

Indeed, Hilbert's program tries to issue proof for the consistency of mathematics. This means the Hilbert's proof, which has requested for the 23 problems and expounded by 1900 congress. Intuitively, it is simple for us to know that 2 + 3 = 3 + 2. One may represent 2, 3, and 5 as II, III, and IIIII, respectively. Hence, one can easily verify the truth of 2 + 3 = 3 + 2 by checking the concatenation of II and III as well as the concatenation of III and II where both amounts are equal to IIIII. Thus, it is not a problematic for finite quantities involved in the equation but it may create problems for infinite quantities.

According to Hilbert in 1925a, the first thing he did was to formalize and axiomatize all of mathematics. This also includes the infinite parts like Cantor's theory of infinite sets. By formalization, one means that every mathematical formula is treated as a string of meaningless logical and mathematical signs, each of them follow one another in line and with definite rules. As Hilbert states: "Hence, the content inference is replaced by manipulation of signs according to rules, and in this way the full transition from a native to a formal treatment is now accomplished" (Hilbert, 1925a. p.381).

After successful formalization, one should make sure that the system does not contradict with itself. To do this, one can inspect the finite formulas and find ways to prove the axiomatic system. Indeed, Hilbert stated: "… a formalized proof, like a numeral, is a concrete and survey-able object. It can be communicated from beginning to end." Our mission is to show 1 ≠ 1 does not exist and it "fundamentally lies within the province of intuition" (ibid. p.383). This means one can use mathematics to inspect or analyze of our proposed system in such cases. This completes what Hilbert believed about proving the consistency of mathematics using safe, finite parts.[1-10]

Critics to Formalism and its Rationale

To the contrary, in 1931 Kurt Gödel introduced incompleteness theorems which state that a consistent axiomatic system does not exist which is formidable enough to reproduce arithmetic and hence Hilbert cannot prove the consistency (Gödel, 1931. p.616) of mathematics by a subsystem of mathematics.

Some in the formalism school suggest that one should consider mathematics as a kind of language. However, if one considers mathematics as a homomorphous language, then one face Gödelian difficulties (Nalimov, 1981). One way to solve the Gödelian problem is the introduction of overriding into the mathematical language with the help of the Bayesian theorem. This means that "one and the same system studied can be described by a variety of mathematical models, all of which have a right to simultaneous existence."[11-15]

After studying mathematical logicism, intuitionism, and formalism, the author suggests that one cannot shape the subject of Mathematics by only one thought of philosophy school without the assistance of other theorems. One of the most recent ideas for Mathematics foundation is by Wilson, 2015 about the "triune philosophy of mathematics"[Figure 1]. Indeed, the author proposes one should continue to work for the nature of mathematics through the collaboration between philosophers and mathematicians.[1] Hence, mathematical education especially philosophy education is important Hong Kong students. The following section will describe the relationship between them among secondary school mathematics lessons.

Mathematics Philosophy and Education

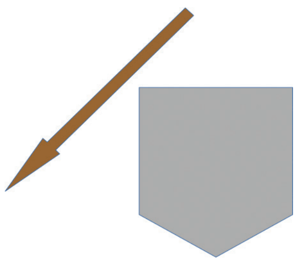
There is a strong connection between mathematics philosophy and education. The author will discuss in detail the use of these philosophy in the teaching of mathematics from both Chinese and Western perspectives.

An Example of Teaching Logicism – proof by Contradiction

There are various examples of using logicism in the teaching of mathematics. One of the most famous instances is using contradiction such as the "joke-proof" by Oscar Perron (1880–1975) who did not have any pedagogical purpose:

"Theorem: 1 is the largest natural number.

Proof: Suppose N is the largest natural numbers, then N^2 cannot exceed N, so N(N-1) = N^2-N is not positive. This means that N-1 is not positive, or that N cannot exceed 1. But N is at least 1. Hence N=1. Q.E.D." (Siu, 2009. p.1)"



The figure shows Mao and Dun

Similarly, one can find plenty of famous paradoxes about contention that exist in both Easnd earn world (Siu, 2009). In the 4th century, B.C.E., Greek philosopher Eubulides of Miletus proposed the well-known Liar Paradox. It is embodied in the terse but intriguing remark "I am a liar." In Eastern China, another philosopher Hon Fei Zi told a popular shield-and-halberd story (Book 15, Section XXXVI, Hon Fei Zi, c.3rd Century B.C.E.):

"My shields are so solid that nothing can penetrate them. My halberds are so sharp that they can penetrate anything."

"How is using your halberds to pierce through your shields?"

The Chinese term "mao dun" or "halberd and shield" means a "contradiction."

Hon told a story to try and show that the Confucianist School of thought in Chinese philosophy was inadequate while the Legalist School was more effective and "superior." He uses *reductio ad absurdum* as his proof (Siu, 2009).[16-20]

Using Intuitionism for Education – recursion and Daoism

Recursion

To illustrate the use of intuitionism in education, the author suggests the use of recursion as an example. Consider the following question from the 29th International Mathematical Olympiad, held in Canberra in 1988 (Siu, 2008):

"Let a and b be positive integers such that ab + 1 divides $a^2 + b^2$. Show that

$[(a^2 + b^2)/(ab +1)]$ is the square of an integer.

One can show the answer to the above problem by using the method of contradiction as presented in Siu, 2008. However, the method does not explain why $[(a^2 + b^2)/(ab +1)]$ must be a square despite confirming that it is so. On the contrary, Siu tries to put $a = N^3$ and $b = N$ so that

$a^2 + b^2 = N^2(N^4 + 1) = N^2(ab + 1)$

I.e. the answer should be in the form (N^3, N, N^2).

Siu formulates a strategy of trying to deduce from $a^2 + b^2 = k(ab + 1)$ the equality $[a – (3b^2 – 3b +1)]^2 + [b – 1]^2 = \{k – [2b – 1]\} \{[a – (3b^2 – 3b + 1)] [b – 1] + 1\}$.

He arrives at the equation k = [(a2 + 1)/(a + 1)] for which a = k =1. By reversing the steps, he would have solved the problem. Indeed, by systematic brute-force checking and looking for some solutions, he has arrived at the partial list as shown below:

a 1 8 27 30 64 112 125 216 240 343 418 512.
b 1 2 3 8 4 30 5 6 27 7 112 8…
k 1 4 9 4 16 4 25 36 9 49 4 64…
By inspecting the pattern, he noticed that for a fixed k, the answers could be obtained recursively as (a_i, b_i, k_i) with $a_i+1 = a_i k_i – b_i$, $b_i+1 = a_i$, $k_i+1 = k_i = k$.
There is a set of "basic solutions" of the form (N2, N, N3) where N ϵ {1, 2, 3 ….}. All other solutions are obtained from a "basic solution" recursively as described above. Therefore
k = [(a2 + 1)/(a + 1)] is a square (Siu, 2008).

Daoism

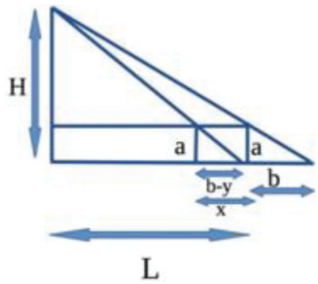
One of the most famous school of Chinese philosophies is Daoism which came into being in the 4th century B.C.E. (Siu, 2008). It was developed as a religion but in this section, the author refers mainly to the philosophical aspects of Daoism. Indeed, the central theme is the Dao (the Way) or the flow of the forces of Nature by which things come together and transform. This reflects a deep-seated Chinese belief that change is a basic characteristic of all things. There are many of authors who have investigated the relationship between Daoism and ancient Chinese science especially in the field of mathematics. One case is the "*Treatise Huainanzi*" which was a Daoist book commissioned by Prince Lui an (179 B.C.E. - 122 B.C.E.).[21-28]
In Essay 3 entitled "Tianwenxun," one finds the following problem on measuring the height of heaven:
"To find the height of heaven (i.e., of the sun) we must set up two 10-che gnomons and measure their shadows on the same day at two places situated exactly 1000 li apart on a north-south line. If the northern one casts a shadow of 2 che in length, the southern one will cast a shadow 1 and 9/10 che long. And for every thousand li southwards the shadow diminishes by one cun. At 20,000 li to the south there will be no shadow at all and that place must be directly beneath the sun. (Thus, beginning with) a shadow of 2 che and a gnomon of 10 che (we find that Southwards) for 1 che of shadow lost we gain 5 che in height (of gnomon). Multiplying, therefore, the number of li to the south by 5, we get 100, 000 li, which is the height of heaven (i.e., of the sun)."
[The translation is adopted from (Needham, 1959).
From a modern perspective, the calculation is explained in the figure below:

y (decrease in length of shadow) is a function of x (distance moved by the gnomon), say y = f(x).
What is x that makes f(x) = 2? That x should be L.
If one knows what f(x) is, then we can calculate L, hence H.
To try and find out what f(x) is, one knows that

$$\frac{a}{b-y}=\frac{H}{L-(x-b+y)} \qquad (1)$$

$$\frac{a}{b}=\frac{H}{b+L} \qquad (2)$$

From (1) and (2), one obtains y = [a/(H–a)] x = αx, where α is a constant.



When x = 1000, y = 0.1. Hence, α = 0.0001, that is, y = 0.0001x. When x = 20, 000, y = 2, so there is no shadow. Hence, L = 20, 000.
H = (b + L) a/b = (2/180 + 20000) (10/2) = 100, 000 + 1/18 (in li)^2
The calculation is based on an over-simplified model of "heaven and earth," so it does not actually measure the "height of heaven." However, the same calculation can be used to measure the height and distance of an inaccessible object. This method of using two gnomons for measurement was explained in detail in Liu Hui's Haidao Suanjing in the 3rd century C.E. (Siu, 2008).

Applying Formalism to Teaching – rules for Calculating the Value of ∏

Consider the following problem (Siu, 1993):
"A circular field has a perimeter of 181 steps and a diameter of 60 and 1/3 steps. What is its area?"
Liu Hiu explained the answer in his commentary, which can be summarized in the following three rules.
Rule 1: The area of an inscribed regular 12-gon equals 3 times the radius times one side of an inscribed regular 6-gon.
Similarly, the area of an inscribed regular 24-gon equals 6 times the radius times one side of an inscribed regular 12-gon.
Rule 2: "The finer one cut, the smaller the leftover; cut after cut until no more cut is possible, then it

coincides with the circle and there is no leftover." This means that the excess of the circle over an inscribed regular polygon will become smaller and smaller as the number of sides is increased, and that the full circle is reached in the limit. The claim is accompanied by a passage which explains how the circle is sandwiched between an inscribed regular polygon and the same polygon with certain added pieces.

Rule 3: Essentially, the general formula relates the area of an inscribed regular $3*2^k$-gon and the perimeter of an inscribed regular $3*2^{k-1}$-gon.

Using the formula in rule 3 and the claim in rule 2, Lui concluded that the area of the circle is half the perimeter times half the diameter (Siu, 2008). To be precise, let A_n, C_n, and a_n denote the area, the perimeter and a side of an inscribed regular n-gon. Rule 1 suggests $A_{12} = 3a_6 r = (C_6/2)$ r and $A_{24} = 6a_{12} r = (C_{12}/2)$ r. In general, we have:

$A_{2m} = (m/2)\ a_m r = (C_m/2)$ r where the values of m are given by $3*2^k$ with k = 1, 2, 3….

Rule 2 suggests A_{2m} tends to A as the limit as m increases indefinitely. Lui Hui gave an estimate for A by noting that $A_{2m} < A < A_{2m} + (A_{2m} – A_m)$.

Rule 3 concludes from $A3*2_k = (C_{3*2}k – 1/2)$ r that in the limit, A = (C/2) r = (C/2) (d/2).

What follows is the famous calculation of ∏ by Lui Hui. He computed a_6, a_{12},…., a_{96} and hence A_{12}, A_{24}…., A_{192} to obtain 314 + 64/625 < A < 314 + 169/625 (with radius equal to 10) [Figures 2-4]. This yield the value 3.14 for ∏.

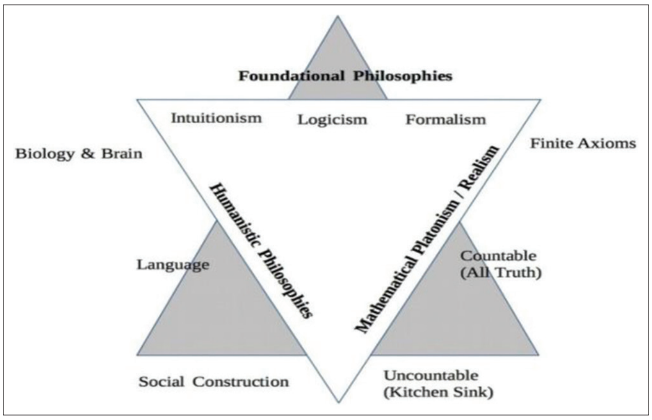


Figure 1: A triune philosophy of mathematics (Wilson, 2015)

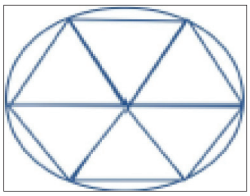


Figure 2: The above figure showed the inscribed "n-gon" (e.g., hexagon) inside a circle. (Siu, 1993)

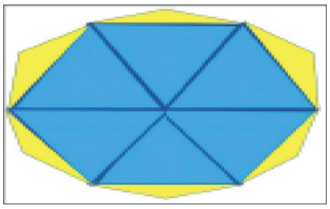


Figure 3: The inscribed 12-gon with area A12 = 3a6r (Rule 1) = (C6/2) r where C6 = (6a6) and a6 is the length of an inscribed hexagon. (Siu,1993)

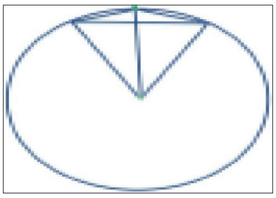


Figure 4: The above figure shown the subdivided and inscribed "2n-gon" (e.g.,12-gon) inside a circle. (Siu, 1993)

DISCUSSION

From the above examples, one can observe that mathematical philosophy can be applied to our everyday teachings of mathematics especially during proof. If one analyses in depth the styles of doing mathematical proof, one finds that there are two types of mathematics (Siu, 2000). These are "dialectic" and "algorithmic" (Henrici, 1974). In general, one can think of dialectic mathematics as a rigorously logical science where "statements are either true or false and objects with specified properties either do or do not exist." (Henrici, 1974. p. 80) It is also an intellectual game played in line with rules about where a consensus will be. It invites meditation and generates mental perception. On the other hand, algorithmic mathematics is a tool for solving problems where one is not just worried about the presence of a mathematical object but also with the indication of its existence. When one is talking the rules of games, it may transform with reference to the urgency of the problem at hand. In addition, it invites action and generates results. (Henrici, 1974. p. 80) It is commonly agreed that a procedural (algorithmic) approach tries to develop more solid ground to build up conceptual understanding. While on the contrary, better conceptual (dialectical) understanding helps us to handle algorithms with more facility, or even to devise improved or new algorithms. One can find a similar case of yin and yang in Chinese philosophy, whereas dialectic and algorithmic complement and supplement each other, each

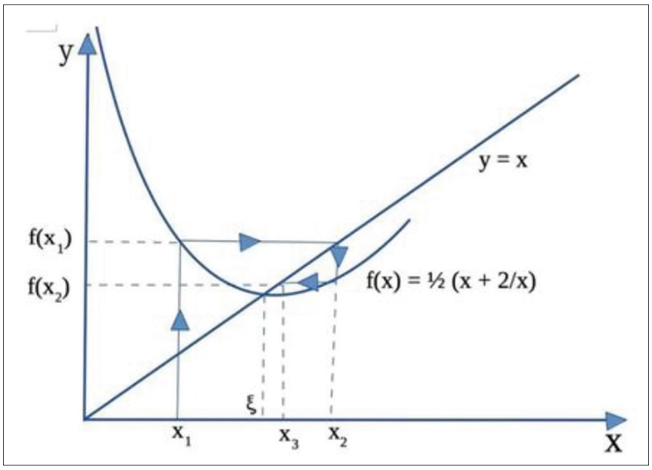


Figure 5: Algorithmic and dialectic in a mathematical proof

containing some part of the other.

When one is asking about the relationship between the above perspectives of doing mathematics and mathematics education, there are several main issues (Siu, 2009). Those issues are: (1) Procedural versus conceptual knowledge; (2) process versus object in learning theory; (3) computer versus non-computer learning environments; (4) "symbolic" versus "geometric" emphasis in learning and teaching; and (5) "Eastern" versus "Western" learners/teachers. According to Anna Sfard, this duality develops it into a deeper model of concept formation through interplay of the "operational" and "structural" phases (Sfard, 1991).

Besides styles in doing mathematics, one should take an in depth look at the performance of students when applying the philosophy of mathematics to daily school environment of Eastern and Western worlds. To cite an instance, consider the above case of "proof by contradiction" mentioned earlier. English mathematician Godfrey Harold Hardy told us that "Reductio ad absurdum," which Euclid loved so much, is one of a mathematician's finest weapons" (Hardy, 1940/1967. p. 94).

Hence, many people believe that "proof by contradiction" is a Western practice and that it is closely related to Greek or even Western culture. Some might question whether Chinese students have an inherent difficulty in learning proof by contradiction since traditional Chinese mathematics does not have such argumentation. However, Siu (2009) finds that most students show learning difficulty in this proving technique regardless of being Chinese or not. This means that there is no relation between a student's cultural background and the proper use of such technique. Indeed, one example of proof by contradiction (where those rules for calculating Π can be viewed as an example of teaching formalism) is by Lui Hui who explains why the ancients were wrong in taking 3 to be the ratio of the perimeter of a circle to its diameter (Siu, 1993. P. 348). Certainly, one does not find the Greek style of "*Reductio ad absurdum*" in ancient China.

Example of using both Algorithmic and Dialectic in a Mathematical Proof: (Siu, 2000) – Find the value for square root of 2. [Figure: 5] Consider the equation: $X^2 - 2 = 0$, one wants to find the value of x through the following algorithmic procedure:

1. Set $x_1 = 1$ and $x_{n+1} = 0.5(x_n + 2/x_n)$ for n≥1.
2. Stop x_n when achieves a specified degree of accuracy. The converging figure shown on the left tells how dialectic mathematics justifies the procedure:

ξ is a root of X = f (X) and ξ is in I = [a, b]

Let f and f'' be continuous on I and |f ' (x)|≤K <1 for all x in I. If x_1 is in I and $x_{n+1} = f(x_n)$ for n≥1, then

Lim $x_n = \xi$

$x_n \longrightarrow \infty$

CONCLUSION

From the above reviews, sample cases and discussions, one can see that mathematics is a part of human endeavor and not just the technical subject as usually taught in our classrooms (Siu, 2008). The author agrees that mathematics has a relationship with the surrounding environment and can be used to describe the natural world. For example, in Daoism, when calculating the height of heaven, it uses the shadow of a long vertical object. Although this method has its deficiencies, it still has some value for the development of modern science. Indeed, Daoism tries to search for the unity between human and nature which is a kind of Taiwan philosophy. It tells us that there are always connection human and nature. Thus, man should follow strictly to the rule of nature so that one can finally achieve the harmony between man and nature. To go ahead a step, we can inherit those wisdom and idea from one generation to another. Moreover, all the previously mentioned examples in mathematical philosophy have a connection to our daily life.

Therefore, as a mathematics teacher, is it important to show these relationships between mathematical proof and daily life to the students? Should teacher point out the connection between mathematical proof and other subjects such as physics and economics to our pupils? The answer is definitely "yes." This is because sooner or later, students will acquire the necessary knowledge and understanding of these relationships through studying other subjects or through other means. If conditions (when they have acquired certain knowledge) are available, teacher should teach students the connectivity between subjects and explain the reasons behind those connections. In such case, mathematics acts as a role of connecting different academic subjects. In other words, it works as a "public servant" to serve other subjects. Usually, it provides a rational instrument for them. Daoism and Lui Hiu which mentioned previously are this kind of human activities examples. They are also forming symbolic representation in Chinese culture which is the same as Jews' one. When one is talking about salvation, one will always refer to the Hebrews' escape from Egypt to Cannan by the guidance of universe's only God. Thus, mathematics is more than a serving subject, it is a cultural one and has recorded down past era's human endeavor.

Mathematics is thus an important subject and it is a part of human culture, it should be used in a moral and ethical way. (Sharygin, 1937–2004) said that

"Learning mathematics builds up our virtues, sharpens our sense of justice and our dignity, strengthens our innate honesty, and our principles. The life of a mathematical society is based on the idea of proof, one of the most highly moral ideas in the world." Siu, (2015: 95) asked:

"Nowadays, how many mathematics teachers still hold on to such belief when they teach?"

The idea being that teachers should explain the ideas behind mathematical proof and use reasons (theories) and examples to link each step of the proof rather than teaching students conceptually. In addition, teachers should teach those cultural contextual behind each topic of the subject.

Indeed, what is the relationship between mathematical proof and philosophy? The answer is: "one can derive philosophical conclusions from philosophical assumptions by mathematical proof. One can build mathematical models in which we can study philosophical problems.[1]" Therefore, as aforementioned, both philosophy and mathematical education such as proofs are essential to Hong Kong secondary students.

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