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# Research and Suggestions in Learning Mathematical Philosophy through Infinity

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## ABSTRACT

Lam, May 2016, explained how mathematics is not only a technical subject but also a cultural one. As such, mathematical proofs and definitions, instead of simply numerical calculations, are essential for students when learning the subject. Hence, there must be a change in Hong Kong's local teachers' pedagogies. This author suggests three alternative ways to teach mathematical philosophy through infinity. These alternatives are as follows: (1) Teach the concept of a limit in formalism through storytelling, (2) use geometry to intuitively learn infinity through constructivism, and (3) implement schematic stages for proof by contradiction. Simultaneously, teachers should also be aware of the difficulties among students in understanding different abstract concepts. These challenges include the following: (1) Struggles with the concept of a limit, (2) mistakes in intuitively computing infinity, and (3) challenges in handling the method of proof by contradiction. By adopting these, alternative approaches can provide the necessary support to pupils trying to comprehend the above mentioned difficult mathematical ideas and ultimately transform students' beliefs (Rolka *et al*., 2007). One can analyze these changed beliefs against the background of conceptual change. According to Davis (2001), "this change implies conceiving of teaching as facilitating, rather than managing learning and changing roles from the sage on the stage to a guide on the side." As a result, Hong Kong's academic results in mathematics should hopefully improve.

**Key words:** Philosophy, infinity, mathematical physics

## INTRODUCTION

There are numerous concerns about the decline in Hong Kong students' performance in mathematics. It is important to identify the reasons for this decline and to find feasible ways to improve the situation. Therefore, it is important to investigate both the teaching and learning processes of mathematics in local classrooms. The purpose of this study is to explore the importance of using mathematical philosophy in Hong Kong's secondary schools. First, the study analyses the academic background of selected teachers. Understanding their mathematical and professional knowledge can help derive suggestions for improvement and enable teachers to understand the difficulties students face in relation to proofs and definitions in theories. Teachers will then be able to more clearly provide suitable assistance to students trying to comprehend relevant mathematical ideas. As a result, there should be an increase in students' mathematics ability. A prime application of this teaching and learning philosophy is through the concept of infinity.

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## LITERATURE REVIEW

First, this study will investigate the academic and professional background of the selected teachers. As described above, recommendations will be made in how to present the concept of mathematical infinity.

### Problem-Based Learning (PBL)

This study suggests that the concept of infinity be introduced through PBL. According to SUNT, winter, 2001, PBL is a method in which students collaborate with their classmates to solve complicated and conclusive problems. PBL can help students develop content knowledge, problem-solving ability, reasoning, communication, and self-assessment skills. In enables students to be interested in the course materials because they are learning necessary skills in a relevant field. PBL keeps learning active and includes important social and contextual factors, which can influence the constructive process (Barrows, 1996; Gijselaers, 1996). In 1996, Wilkerson and Gijselaers claimed that the main characteristic of PBL is that it is student centered, while teachers act only as facilitators rather than disseminators.

Teachers use open-ended problems (or ill-structured problems) as an initial stimulus and framework for students' learning (Wilkerson and Gijselaers, 1996). The main duty of teachers is to develop students' interest in a subject. Rather than recalling information, pupils are required to learn on their own through group work and become self-directed learners (SUNT, winter, 2001).

In practice, teachers can introduce the concept of infinity through the following open-ended questions:

OPEQ1: Consider a frog jumping towards the edge of the pond, each time halving the distance it jumps. Can it reach the edge of the pond?

OPEQ2: Think carefully about Gabriel's Wedding Cake, with its elaborate circular layers. Can the top and sides of all the layers be frosted?

OPEQ3: Can a quadrilateral be cyclic when the sum of the pairs of opposite angles is 180°?

Lam, August 2016, provides various suggestions of examples to use when introducing the concept of infinity. Some of those suggestions will be explored in the following section.

## Recommendations for teaching pedagogy

### *Teaching the concept of limit in formalism using story telling*

One of the most famous ways to teach the concept of infinity is using calculus, which allows formal instruction on the concept of a limit. First, a teacher should recount the following story for primary students:

"A frog is sitting in the middle of a 4-m-diameter circular pond, on a lily pad.

It jumps 1 m toward the edge of pond (onto another lily pad) and keeps jumping towards the edge. However, the frog gets more and more tired after each successive jump and only jumps half the distance of the previous jump. It does not take much thought to figure out that the frog never reaches the edge in any finite amount of time, assuming the jumping time (including the time between jumps) is constant.

"Now assume that as the frog nears the edge of the pond, it gets more and more excited about reaching the edge, and on each successive jump, it doubles the jump rate (including the time between jumps). However, its legs still get tired, so each successive jump is only half the distance of the preceding one.[1]

"Can the frog reach the edge of the pond?"

The above philosophical story can be viewed as a paradox of motion. Lam, August, 2016, shows detailed mathematical knowledge of the concept of a limit is required of teachers. Suggestions for teaching pedagogy are as follows (Liang, 2016):

1. Recognition of misconceptions: In class, teachers can ask students to solve and explain the problem of the frog reaching the edge. Obviously, different students will give different solutions and explanations. Based on their statements and understanding, teachers can then categorize the character of them is conceptions.
2. Clarification: Under the supervision of teachers, students are encouraged to discuss the pros and cons of the proposed results for the problem. After debate among the students, the proposed results survive is eliminated or remain undecided. Teachers must then summarize the misconceptions, with some being unique and others more common.
3. Confront misconceptions: Teachers can point out students' illogical responses and provide counter-examples to address their misconceptions of the problem. The formal concept should make sense.
4. Accommodation: Students are finally willing to accept the new knowledge of a limit and change their misconceptions.

However, there are several limitations to the above conceptual conflict strategy (Liang, 2016). First, it is difficult for individuals to reject their old ways of thinking.

Second, the strategy is not revolutionary in reconstructing students' ways of thinking. Third, it cannot be applied to all teaching settings (Liang, 2016). Although there are critics of this strategy, this author still believes that by discovering the misconceptions surrounding the idea of a limit through discussion, teachers can help students confront their challenges. This leads to a conceptual change, and students acquire new knowledge.

After discussing the pedagogy for teaching the concept of limit in formalism, this paper proceeds to recommendations for intuitively teaching infinity computation.

***Using geometry to intuitively learn infinity through constructivism***

Teachers can constructively introduce Gabriel's Wedding Cake to junior secondary school students through the guided activity presented below2. It is important to note that an activity is said to be of a constructive instructional design if it fulfills the criteria (Murphy, 1997a and Jonassen, 1994: p.35) described as follows:

1. There are multiple representations of reality.
2. The complexity of the natural world is shown.
3. Knowledge can be constructed rather than reproduced.
4. Tasks avoid abstract instructions and instead present contextualized instructions.
5. Teachers provide real-world, case-based learning environments, and prevent predetermined instructional sequences.
6. Student reflection is encouraged.
7. Content and context-dependent knowledge is constructed.
8. Knowledge is constructed collaboratively from social negotiation.

The results of Gabriel's Wedding Cake lead to a famous and beautiful paradox: While the volume of the cake is finite, one cannot frost the cake. As a result, a cognitive conflict occurs among students, and it is up to teachers to aid in solving the problem. To do so, teachers must do the following (Sayce, 2010):

1. Lead the discussion: Teachers must use an effective strategy for managing discussions (Swan, 2005). Teachers should prepare open and probing questions in advance but be ready for unexpected responses to questions.
2. Keep up motivation: Teachers must maintain students' motivation in light of the conflict raised. Thus, teachers should praise learners for a variety of reasons. It is necessary to remove the fixation away from getting the right answer and refocus on the thinking itself (de Geest, 2007). Learners must value the process of learning, and in addition, teachers' praise is helpful to those who are underachieving (Boaler, 2009) or for those with a fixed intelligence belief and fear failure (Dweck, 2000).
3. Prepare learners for the conflict: Since the work is an emotional experience, it is helpful for learners and teachers to be encouraged in working in this way and keep up enthusiasm. Although they might not make much progress initially or be considered to "not be doing proper work," this method is still an effective way to learn. On the other hand, if students cannot always obtain a definitive answer, they might become de-motivated and feel as if they are "learning nothing."
4. Provide a common language: A common language is needed between teachers and learners to be able to describe, explain, and discuss the problem and listen to each other.

If the two sides of the conflict cannot be connected, then students can actively participate in finding reasons for the cognitive dissonance. Certainly, there are critics of the above-proposed resolution teaching theory; however, this author believes that teachers should maintain students' motivation for addressing the conflict and lead a discussion before arriving at a conclusion. Language is a determining factor of the successful implementation of this philosophy.

***Schematic stages for proof by contradiction***

To teach using a dynamic geometry environment (DGE), a teacher must create a cognitive conflict for students (Arshavsky). During a lesson, a teacher should show students a quadrilateral with its vertices dragged around (Fujita *et al*., 2007). Students should be asked characterize and define the figure and to justify their conclusion. The categorization of the results will vary considerably when the students try to draw the collinear of the three vertices of the quadrilateral or the intersection of the two sides. This activity produces cognitive conflict, which can then be used as a starting point for the discussion of the role of definitions in mathematics (Fujita *et al*., 2007). Lopez *et al*. developed the "Argumentative Stages of a Proof by Contradiction in DGE," which is described as follows (Lopez *et al*., 2002):

Initial Argumentative Stage – Construction of a biased dynamic geometry micro-world.

1. "A" (e.g., a quadrilateral ABCD) – a type of geometrical configuration in DGE

2. Impose condition “C(A)” (e.g., interior opposite angles are supplementary) on “A”
3. A biased DGE labeling with a forced presupposition C(A) (e.g., the arbitrary labeling of∠DAB=2a and ∠DCB=180-2a) Second Argumentative Stage – Construction of a pseudo-object
4. Observation guided by geometrical intuition: a kind of hybrid state between the visual-true DGE and a pseudo-true rationale of “C(A)”
5. Construction of a pseudo-object “O(A)” that inherits internal inconsistency (e.g., a quadrilateral EBFD)Third Argumentative Stage – Discovery of a locus of validity
6. Employ the drag-until-vanish strategy on “O (A):” When part of A is being dragged to different positions, “O(A)” might vanish (i.e., a plane figure to a line, a line to a point).
7. Discovery of a locus of validity associated with “O(A),” where the biased micro-world is realized in the Euclidean world

Final Argumentative Stage – Make conjecture and organize a proof by contradiction.

The pedagogy presented above represents a way to teach proof by contradiction in a step-by-step approach. According to H and J’s Reduction ad Absurdum proof, one may consider the behavior of the pseudo-quadrilateral as a meta-pattern found among patterns. Hence, interactions between the person and the DGE are a determining factor of the successful achievement of insight and understanding. This author believes that students can handle this form of interplay in a better and more cognitive way through the above schematic stages of argumentative activity.

## Proposed research methodology

There are numerous concerns about why and how to apply philosophy when teaching mathematics to Hong Kong secondary school students. Therefore, an empirical case study is performed to answer these questions.

## Research strategy – case study

There are many strategies that can be implemented in studying social science, including case studies, surveys and histories (Chan, 2009). Case studies can be adopted to bridge the gap between experiments and theories concerning the use of mathematical philosophy in daily lessons. In this study, case studies are used to examine the understanding of infinity among secondary students by addressing two basic research questions:

RQ 1: Why should there be a shift to philosophical proof in our Hong Kong secondary school curriculum?

RQ 2: How should teachers introduce the concept of infinity through different pedagogies? Answering these questions can improve the understanding of the importance of philosophy, especially in the field of mathematics education. Moreover, Yin, 1994 states:

“A case study is an empirical inquiry that investigates a contemporary phenomenon within its real- life context, especially when the boundaries between phenomenon and context are not clearly evident” (p.13).

In general terms, a case study is a naturalistic inquiry for a large setting-specific instance (Chan, 2009). At the same time, it is open to findings that do not have predetermined constraints. Anthropologists sometimes label a case study as a “thick description” since it can be referred to as a delineated event (Patton, 2002; Charles and Merter, 2002). As mentioned by Yin (1994):

“In general, case studies are the preferred strategy when ‘how’ or ‘why’ questions are being posed, when the investigator has little control over events, and when the focus is on a contemporary phenomenon within some real-life context” (p.1).

The present research satisfies all of the aforementioned requirements for case studies:

1. It tells us why and how philosophy should be included in the Hong Kong secondary school mathematics curriculum.
2. It proposes several examples for how mathematics should be taught from a philosophical perspective. However, ordinary mathematics teachers might have their own methods, that is, teaching in a naturalistic setting. If this is the case, then this study does not overlook their teaching strategies (Chan, 2009).
3. It involves contemporary events rather than historical events since the method of teaching and learning the concept of infinity can be investigated and observed directly (Chan, 2009).

## Empirical study

What is an empirical study? According to Goodwin, 2005, a study that adopts empirical evidence is an empirical study; that is, it obtains knowledge by direct or indirect experience or observation. In scientific methods, “empirical” can describe a working hypothesis that can be tested through observation and experimentation. This evidence can be evaluated either qualitatively or quantitatively. The answer to empirical questions proposed by a researcher can be concluded from well-defined and justified collected testimony (Goodwin, 2005). A real-case scenario includes the following “empirical research cycle,” as suggested by A.D. de Groot:

1. Observation: Having well-collected and organized empirical facts that form the hypothesis requires collecting data through scientific observation or experimentation (Kosso, 2011). Observation can be performed qualitatively, whereby the existence or absence of a property is noted, or quantitatively, whereby a numerical value can be attached to the observed phenomenon (Kosso, 2011).
2. Induction: A hypothesis is formulated. Inductive reasoning is the process in which various premises are considered as providing strong proof about the truth of the conclusion (Copi *et al*., 2007). From a philosophical perspective, these premises suggest a truth without ensuring the conclusion (Audi, 1999). Thus, it is easy to transform general statements into an example.
3. Deduction: The consequences of the hypothesis as testable predictions can be determined. As opposed to inductive reasoning, deductive logic is the process of reasoning from one or more statements or premises such that a logical conclusion can be reached (Sternberg, 2009). In deductive reasoning, general rules are applied over a closed domain of discourse, narrowing the range under consideration and thus reductively arriving at a conclusion.
4. Testing: The hypothesis is tested using the new empirical data. In general, an experiment is a procedure undertaken to refute, verify, or validate a hypothesis (Stohr-Hunt, 1996). Experiments attempt to identify causes and effects by showing the outcome when a factor is manipulated. There can be great variation in the goal and scale of experiments, but they all rely on repeatable procedures and logical analyses of the results (Stohr-Hunt,1996).
5. Evaluation: The test outcomes are evaluated. There is a set of standards that govern the systematic determination of a subject’s significance, merit, and worth (Staff, 1995). Evaluation helps in decision-making processes across different fields, such as in organizations. Furthermore, it enables reflection and identification of future change (Staff, 1995).

## Research design

The section highlights the basic structure of empirical research, which comprises observation, induction, deduction, testing, and evaluation. The research design for each category is described below:

## Observation design

This study will recruit secondary school students by sending invitational letters to all day schools in Hong Kong. The research will last for one academic year (Lau, 2009). Participants will be students enrolled in primary years 1–6 and secondary years 1–5 and will be under the supervision of the teachers and principals. There will be class observations at the chosen schools to determine how teachers introduce the concept of infinity. The school visitation period will last for 1 year. Throughout this period, data will be regularly collected for research evaluation. The main purpose of the study is to discuss how teachers should introduce the concept of infinity to primary and secondary school students. Data will be obtained from school visits, during which students’ and teachers’ perceptions will be recorded. Deduction is top-down logic, while induction is bottom-up logic.[1-15]

## Induction design

After school visits and class observations have been conducted, this study will test the following hypotheses to determine how students learn the concept of a limit:

$H_0$: Most Hong Kong students do not fully understand the concept of a limit.

$H_1$: Hong Kong students can understand the concept of limit.

In addition, another inductive hypothesis for students learning about “contradiction:”

$H_0$: Most Hong Kong students show difficulties in applying the technique of contradiction to mathematical proofs.

$H_1$: Students can apply the technique of contradiction to mathematical proofs very well. The final hypothesis from the class observations of students' intuition is as follows:
$H_0$: There are always errors that exist in students' intuition when they learn about things related to infinity.
$H_1$: There are no errors that exist in students' intuition when they learn about things related to infinity.

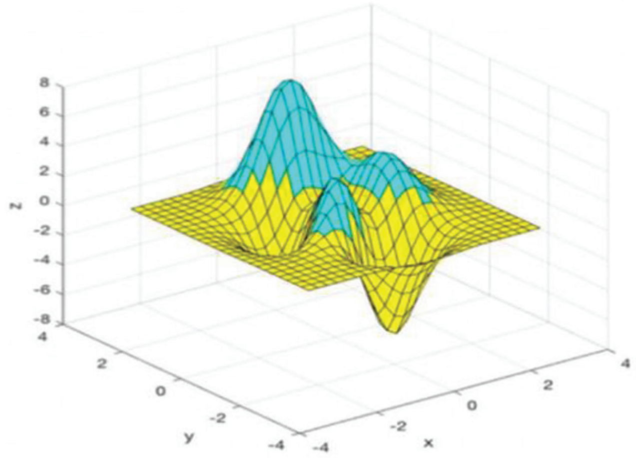
**Deduction**

$D_1$: If students have difficulty understanding the concept of a limit, then they will struggle when trying to comprehend the idea of infinity.
$D_2$: If students have difficulty applying the technique of proof by contradiction, then they will be unable to understand the concept of logicism in philosophy.
$D_3$: If students intuitively make mistakes when learning about things related to infinity, then they may lack an understanding of in the relevant mathematical theory.

## TESTING – QUASI-EXPERIMENTAL STUDY

To test the hypotheses, this study will use the quasi-experiment method. Specifically, the investigation of the students' concept of infinity will be conducted using a computerized Taylor polynomial approximation study (Kidron and Tall, 2014). The software MATHLAB will be used in this experiment. The aim is to determine the "convergence of sequence of functions visually considered as graphs that converge onto the limit function" (Kidron and Tall, 2014: p.1). "The approach offered in this study stimulated explicit discussion not only of the relationship between the potential infinity of the process and the actual infinity of the limit but also of the transition from the Taylor polynomials as approximations to a desired accuracy toward the formal definition of limit" (Kidron and Tall, 2014: p.1).[11-20] For example, one may expand a polynomial f(x) around "0" for sin(x) up to degree 5 as described below:
$P^5(x) = x – x3/3! + x5/5! +….$
The error can be expressed as a function of x and c. Indeed, it is given by $f(x) – P^5(x)$, where the Lagrange remainder is:
$(f^{(6)}(c)\, x^6)/6!$ for some c between 0 and x.

Therefore, the absolute value of the error can be plotted as follows:




In this quasi-experiment, the researcher will analyze how students can alter their understanding from a symbolic and embodied world to a formal definition of limit. The first group of students will be required to find a polynomial with a given degree.
Polynomials obtain the highest possible order of contact with a given function. Students will be guided in analyzing the process of convergence and describe what they have observed through dynamic graphical animation. Finally, they will be asked to translate dynamic pictures into analytical language. Hence, students can change the parameters and choose different functions to construct their own functions. For example, they can test the convergence of the Lagrange Remainder for different functions.
Simultaneously, another group of students will be involved in the approximation process following the original text of Euler (1988). They will use MATH-LAB for the "continued division procedure" to perform the calculation for function 1/(1-x), as stated by Euler. The aim of this experiment is to use MATH-LAB commands to carry out Euler's algorithmic thinking. This will allow insight into Euler's "development of functions in infinite series."
The final group of students will be a control group. They will be presented with the concept of a limit via traditional and formal instructional methods. In addition, questions and answers and whole class discussions will be employed in the control group (Narli, 2011).

## EVALUATION

There will be a class discussion for participating students in which function sin(x) and Taylor polynomial: $P_n(x) = a_0 + a_1x + a_2X^2 + a_3x^3 + \ldots + a_nx^n$ are both known, as is computation error: $f(x) - P_n(x)$. In addition to the class discussion, a written test will be administrated in an effort to analyze the personal conceptions of individual students (Kidron and Tall, 2014).[21-30]

## Research data

There are two categories of research methods: Qualitative and quantitative. The selection of a method depends on the types of research questions. The aim of the methods is to construct knowledge, and they can be implemented in a mixed and complementary way (i.e., the mixed- method approach).

This study employs an inductive process for exploring issues and investigating phenomena related to teaching and learning the concept of infinity. Therefore, a qualitative method is chosen for this research. Quantitative data are included as the following:

1. There will be 180 students participating in the focus group discussions (10 within each of the three phases per grade and in a total of three schools).
2. There will be 36 teachers participating in the interviews (two within each of the three phases per grade and in a total of three schools).
3. There will be field notes and video recordings taken in 18 class observations (one within each of three phases per grade in a total of three schools).
4. There will be open-ended questions and written tests administered after each class observation (out of a total of 540students).

Qualitative data analysis will be used for the examination of the open-ended questions, focus group discussions, interviews, field notes, and video recordings. In 1987, Strauss outlined the coding steps as open coding, axial coding, and writing memos to find the main themes. This study will investigate how philosophy education can support the teaching and learning of infinity.

Students' written tests will be used to examine the concept of infinity after each class observation. This will enable the evaluation of the significance of philosophy education in normal lessons. As a result, the necessary alterations can be made to the present Hong Kong secondary school mathematics curriculum.

There are a total of three phases in this study: A development stage, a school visit stage, and a data analysis stage. In the development stage, invitational letters, experiments, and written tests will be developed and prepared for implementation. For the school visit stage, students from three different ranking schools in primary grades 1 to 6 and secondary grades 1 to 5 will be interviewed and observed. In the data analysis stage, information will be qualitatively analyzed.

When questioning the validity and reliability of the above qualitative research, one may first refer to its internal and external validity (Yung, 2011). In 1999, Huitt *et al*. defined internal validity as "the confidence in the change of a dependent variable which is caused by change in an independent variable." External validity is considered the extent of to which a study's results can be generalized. The present research is strong in its internal validity but weak in its external validity.

This is due to the goal of the study, which is to understand the phenomenon rather than trying to present a generalized result. In addition, only a limited number of people, situations, and events are included in the research because of constraints on time, labor, and resources. Detailed field notes and audio-visual recordings of the class observations and interviews will enhance the research reliability (Yung, 2011). After viewing the examples of teaching infinity through mathematical philosophy, one may ask what the outcomes and value of the current study will be.[31,32]

## Difficulties students face when learning infinity

### *Struggles with the concept limit*

As mentioned in the previous section, the definition of a limit is abstract. Most students may have difficulty understanding the concept of a limit (Swinyard and Larsen, 2012):

$\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$,

Such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$

According to Fernandez in 2004, students are confused about the following:

i) What do ε and δ really represent?
ii) What the relationships between those variables and parameters are as described in the definition?

iii) Why does |x – a| need to be positive while |f(x) – L| can be either positive or negative?

1. The action of evaluating f at a single point x that is considered to be close to, or even equal to, a.
2. The action of evaluating the function f at a few points, each successive point closer to a than was the previous point.
3. Construction of a coordinated schema as follows.
 a. Interiorization of the action of Step 2 to construct a domain process in which x approaches a.
 b. Construction of a range process in which y approaches L.
 c. Coordination of (a), (b) via f. That is, the function f is applied to the process of x approaching a to obtain the process of f (x) approaching L.
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of Step 3 is encapsulated to become an object.
5. Reconstruct the processes of Step 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, In symbols, 0 < |x – a| < δ and | f (x) – L| < ε.
6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.
7. A completed ε – δ conception applied to a specific situation.

The reason for students' misconceptions is their struggles with experiencing the quantification of the definition of a limit (Cottrill *et al*., 1996; Dubinsky *et al*., 1988; Tall and Vinner, 1981). Cottrill, 1996, decomposed the concept of a limit and how the human brain might construct it, in other words, what students might construct during the process of trying to comprehend the concept of a limit (Swinyard and Larsen, 2012). Figure 3 lists the ways to decompose the concept of a limit. To conclude, it could be said that understanding mathematical concepts such as a limit is a sophisticated process. An individual might always struggle with the quantification of its ε and δ definitions. To address this problem, an individual may be able to learn this concept through graphical representation, numerical experimentation, or related theorems. Hence, students may "reinvent" a new definition of a limit to understand the concept more clearly, which, in turn, might help them overcome the complex cognitive barriers formed by the original definition.

1. The action of evaluating f at a single point x that is close to, or even equal to, a.
2. The action of evaluating function f at a few points, with each successive point becoming closer to a than the previous point was.
3. Construction of a coordinated schema as follows.
   a) Interiorization of the action of Step 2 to construct a domain process in which x approaches.
   b) Construction of a range process in which y approaches L.
   c) Coordination of (a), (b) via f. That is, function f is applied to the process of x approaching a to obtain the process of f (x) approaching L.
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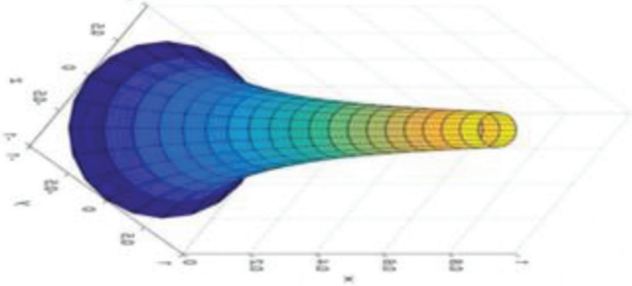
7. A completed ε – δ conception applied to a specific situation.

***Mistakes in intuitively computed infinity***

In the previous section, a gap was revealed between intuitive computation of the famous solid Gabriel's trumpet and formal integration. Students usually make mistakes in intuitively computing infinity. How and why is there such a gap? To answer this question, one must first understand the results from the formal integration of Gabriel's trumpet:

Volume of Gabriel's trumpet: When one rotates the graph of function f(x) = 1/x around the x-axis and computes the volume between 1 and ∞n one will have:

$$\lim_{R\to\infty}\left(\int_1^R \frac{\pi}{x^\wedge 2}\right) dx = \underset{R\to\infty}{lim}\left[\frac{-\pi}{x}\right]_1^R = \underset{R\to\infty}{lim}\left[\frac{-\pi}{R}\right]+\pi$$



Gabriel trumpet generated using Math-lab (rotating about x-axis).

Clearly, the above limit converges to ∏o The cross-sectional area for integration is $\prod/x^2$, where R is a fixed value.

***Surface area of Gabriel's trumpet***

$$\lim_{R\to\infty}\left(\int_1^R \frac{2\pi}{x}\right) = 2\pi \lim_{R\to\infty}[\log R]-[\log 1]$$

Note that the surface area of the small slice is larger than (2aner dx. Therefore, the total surface area is larger than the above limit and is ∞s

Formal computation tells us that it is impossible to paint the surface area of Gabriel's trumpet. Most students intuitively think that one can use a finite amount of paint. Wijeratne, 2015, suggests that this is the result of the intuitive rule "same A-same B." In 1999, Stavy and Tirosh explained that if there are two equal systems with a certain quantity A but with another different quantity B, students often argue that the "same amount of A implying the same amount of B." This finding shows that the alternative misconception comes from the above common, intuitive rule.

Therefore, it is teachers' role in placing emphasis to students when the "same A-same B" rule should be applied appropriately.

"The differences between these two types of situations should be discussed, stressing the inapplicability of the intuitive rule in the second. This could help students to form the application boundaries of the intuitive rules. In addition, we recommend that students should be encouraged to criticize and test their own responses, relying on scientific, formal knowledge" (Stavy and Tirosh, 1999: p.64).

***Puzzles in handling the proof-by-contradiction method***

As mentioned, students may have difficulties in applying the proof-by-contradiction method. According to Antonini and Mariotti in 2008, there are three puzzles in this method:

1. The first puzzle is when to use proof by contradiction. Most mathematicians believe that there are two criteria for using the method: "(1) The given conditions are not able or not easy to be manipulated; and (2) the negation of conclusion reveal an obvious representation within a familiar system" (Lin and Lee, 2016:p. 4–443).
2. The second puzzle is how to connect the contradiction and the principal statement. Given principal statement p, one must give a direct proof to its negating (secondary) statement. The first sub-puzzle is how students should formulate the proof for negating statement q. After the proof of statement q, the conclusion constitutes a contradiction to q. Thus, principal statement p is correct. This means "if ~q then ~p" implies "if p then q" (Lin and Lee, 2016:p. 4–443). Hence, the second sub-puzzle lies in how students should link the final proof of statement q to principal statement p.
3. The third puzzle is what should be done to treat impossible mathematical objects created during the proof of statement q. There is a need to discard these mathematical objects after the proof. Thus, one can use the adductive process for mobilizing explanatory hypotheses.

Teachers are required to fill the gap for students between the contradiction and what the statement must prove. One may conclude the following:

"If such indirect proofs are encouraged and handled informally, then when students study the topic more formally, teachers will be in a position to develop links between this informal language and the more formal indirect-proof structure." (Thompson 1996, p.480)

After discussing several cognitive models regarding students' learning in mathematical philosophy, this study has revealed that reasoning is a significant factor for students to achieve academic success in mathematics. Thus, there is a need to promote students' focus on understanding mathematical proofs and definitions rather than only on performing numerical calculation.

## CONCLUSION

Students' understanding of mathematical proofs and definitions is essential in raising Hong Kong's mathematical academic standards. Changes must be made to the teaching pedagogies of local education. At the same time, teachers should have their own ideas about how students might be able to learn abstract mathematical concepts. In doing so, teachers can provide the necessary support for pupils trying to comprehend those concepts, which, in turn, can lead to changes in students' beliefs. This progress can then be analyzed through conceptual transformations, or "the kind of learning required when the new information to be learned comes in conflict with the learners' prior knowledge is usually acquired on the basis of everyday experiences"

(Vosnadiou and Lieven, 2004, p. 445). There are four criteria for beliefs: Lived experience, belief rejection, belief replacement, and synthetic model (Rolka *et al*., 2007). In 1997, Appleton developed a model for describing and analyzing students' learning:

1. The new information creates an apparent identical fit with the students "present ideas."
2. The new information forms an approximate fit with the students' existing concept, which seems to be related, but the details are unclear.
3. The new information is only acknowledged

without an explanation by the ideas attempted so far. This is known as an incomplete fit.

Similarly, teachers' beliefs about mathematics are as follows (Liljedhal, 2006):

1. Toolbox: Mathematics is about numbers and rules.
2. System: Mathematics is the science of pattern and structure.
3. Utility: Mathematics is all around us. We live in a quantitative society.
4. Process: Mathematics is a construction of understanding that individuals build.

Gradually, teachers have changed from system and utility points of view to a process standpoint. This evolution can be interpreted as "unlearning the process of learning to teach mathematics better" (Liljedhal, 2006: p.327).

From the above, it is apparent that the teacher's role has changed from a sage on the stage to a guide on the side (Davis, 2001). In other words, teachers have transformed to now act as facilitators. Indeed, what is said to be the best relationship between teaching and learning is (Siu, 2014: p.4) as follows:

**"**Teaching and learning help each other;" as it is said in the Charge to Yueh, "Teaching is half of learning."

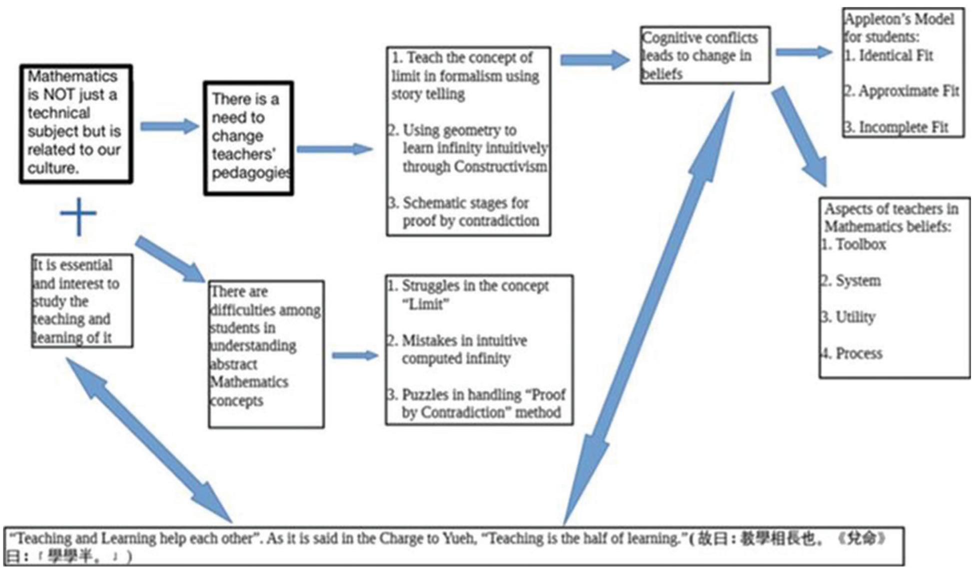
## Remarks

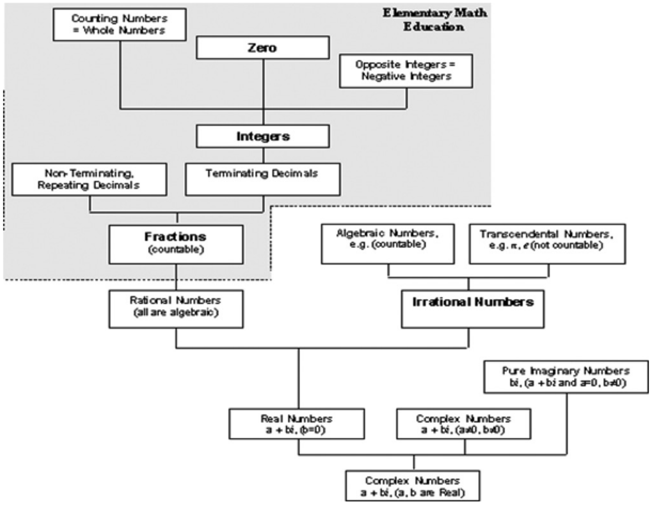
This author notes that one may further extend the present Taylor series from the infinitesimal view (Stewart and David, 2015). The idea comes from the "arbitrarily small" quantities dx and dy in the historical development of calculus. The prescribed Taylor series in this paper can be handled in the field R((x)) consisting of power series with a finite number of negative powers. However, if there is a sequence $a_1, a_2, \ldots, a_n$ where a function a: N —> R with $a(n) = a_n$, what would be the appropriate extension field? Obviously, this would be no problem for Leibniz, since his work fits well with infinitesimals. Thus, to fulfill sequence a(n), one must extend the normal mathematical analysis into non-standard analysis. It extends the real numbers R into R* (the super ordered field), which is called the hyperreals (Stewart and Tall, 2015). This author believes that it is one of the structures that stays between natural and real numbers because R* is the infinitesimal treatment of real numbers but does not belong to the complex number.

To conclude, the full picture of number systems is depicted in the following table:

| | |
|---|---|
| Countable Sets | Natural numbers, integers, rational numbers, constructible numbers, algebraic numbers, periods, computable numbers, definable real numbers, arithmetical numbers, Gaussian integers |
| Division Algebras | Real numbers, complex numbers, quaternions, octonions |
| Split Composition Algebras | Over R split complex numbers, split quaternions, split octonions over C bi-complex numbers, bi-quaternions, bi-octonions |
| Other Hypercomplex | Dual numbers, dual quaternions, hyperbolic quaternions, sedenions, split- bi-quaternions, multicomplex numbers |
| Other Types | Cardinal numbers, irrational numbers, fuzzy numbers, hyper-real numbers, Levi-Civita field, surreal numbers, transcendental numbers, ordinal numbers, p-acid numbers, super-natural numbers, super-real numbers |

The figure below shows part of the structure of the number system (from Google Images):






The Conceptual Framework of the Present Study (Yang, 1989).

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