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Analytical Solutions of the Camassa Holm Degasperis Procesi Equation and Phase Plane Analysis

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ABSTRACT

In this work, we study the Camassa Holm Degasperis Procesi equation through solving it by three different methods which are the factorization technique, the Cole-Hopf method, and the Schwarzian derivatives method, plus studying the stability of the system through the Phase Portrait method**.** Illustrations of the solution are presented using symbolic software which shows different pattern formations.

Key words: Camassa-Holm-Degasperis-Procesi equation, convection, factorization technique, cole-hopf transformation, schwarzian derivative, phase portrait.

INTRODUCTION

The following family of evolutionary 1 + 1 PDEs that describes the balance between convection and stretching for small viscosity in the dynamics of one-dimensional nonlinear waves in fluids was investigated in Holm and Staley.[1]

$$m_t + u\, m_x + b\, u_x\, m = \nu\, m_{xx} \tag{1}$$

Where u is the velocity, $m = u - \alpha^2 u_{xx}$ is the momentum density and b is a balance parameter that represents the ratio of stretching to convection. Such that $u\, m_x$ is the convection term, $b\, u_x\, m$ is the stretching term and $\nu\, m_x$ represents viscosity where ν is the kinematic viscosity. It was found that eq. (1) for any $b\neq1$ is included in the family of shallow water equations [2]. It is very well known that Camassa and Holm derived an equation, which is a completely integrable Hamiltonian system, for unidirectional motion of shallow water waves in a particular Galilean frame (CH equation)[3] as an approximation to the Euler equations of hydrodynamics.

$$m_t + u\, m_x + 2\, u_x m = -c_0\, u_x - \gamma\, u_{xxx} \tag{2}$$

Where the constants α^2 and γ/c_0 are squares of length scales and $c_0 = \sqrt{g\, h_0}$ is the linear wave speed for undisturbed water of depth h at rest under gravity g at spatial infinity.[4] In the previous equation, the RHS represents dispersion and the value of the balance parameter was taken as $b = 2$. The Camassa-Holm-Degasperis-Procesi (CHDP) was introduced by Degasperis and Procesi [4], also studied in [5]

$$u_t - c_0 u_x + (b+1)\, u\, u_x - \alpha^2 (u_{xxt} + u\, u_{xxx} + b\, u_x u_{xx}) + \gamma\, u_{xxx} = 0 \tag{3}$$

In case of b=3, γ=0, α=1 and c_0=0 eq. (3) becomes the Degasperis procesi (DP) equation. The importance of the CH and DP equations is due to their relevance to the modeling of wave breaking, which is one of the most important and mathematically elusive phenomena in the study of water waves [6]. The two equations posses stronger nonlinear effects than both of the Korteweg–de Vries and Benjamin–Bona–Mahoney equations. The relevance of the two equations, CH and DP equations, as models for the shallow water wave propagation had been proved in addition to proving that both equations are valid approximations to the governing equations for water waves. The approximations were; propagation in one direction over a flat bottom with no viscosity, no shear

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stress and no compressibility under the influence of gravity and surface tension.

The CH-DP equation was studied in Mhlanga and Khalique[7] using Lie symmetry method along with the simplest equation method to derive its exact solutions. These solutions were plotted showing a solitary wave behavior. A method for the classification of all weak traveling-wave solutions was presented Lenells[8] and applied to the CH and DP equations showing the existence of smooth, peaked, and cusped traveling-wave solutions plus showing fractal-like wave profiles. Integrable equations with second-order Lax pair like (CH) exhibit interesting conformal properties and can be written in terms of the so-called conformal invariants (Schwarz form).[9] The squared eigenfunctions of the spectral problem, associated to the Camassa-Holm equation represent a complete basis of functions, which helps to describe the Inverse Scattering Transform (IST) for the Camassa-Holm hierarchy as a Generalized Fourier Transform (GFT).[9] Using GFT author described in Ivanov[9] explicitly some members of the CH hierarchy, including integrable deformations for the CH equation. Furthermore, he showed that solutions of some 2+1-dimensional generalizations of CH can be constructed via the IST for the CH hierarchy. A generalization for the rational exponential method was presented and applied to the Vakhnenko equation or the reduced Ostrovsky equation, and the modified CH and DP were discussed in Nuseir[10] which lead to exact solutions for these equations were found. Periodic loop solutions of the CH-DP equation were investigated using the dynamical system theory.[11] Authors studied the bifurcation and global behavior of the equation and obtained the conditions under which the periodic loop waves appear and their representations be obtained. Plus, they derived explicit analytical periodic loop solutions of the same equation. The CH-DP equation was investigated in Xie and Wang[12] using the bifurcation method of planar systems and the simulation method of differential equations. The bifurcation phase portraits are drawn in different regions of parameter plane. The planar graphs of compactons and generalized kink waves are simulated. Exact explicit parameter expressions of compactons and implicit expressions of generalized kink wave solutions are given, and the dynamic characters of these solutions are investigated. The method of dynamical systems was used in Li and Zhang[13] in addition to studying the traveling wave solutions of a special CH-DP equation. Exact explicit parametric representations of smooth solitary waves, solitary cusp waves, breaking waves, and uncountably infinitely many smooth periodic wave solutions are given. In different regions of the parametric plane, different phase portraits are determined. The so-called loop soliton solution is discussed. A general modified CH-DP equation was suggested in Fan *et al*.[14] and traveling wave solutions were developed. The bifurcation and global behavior of CH-DP equation were studied in Xie and Du.[15] Authors obtained exact solutions such as periodic wave, solitary wave, cusp wave, loop, and kink waves and made their numerical simulations. Plus, they have studied the singular points. Using the dynamical system method they have found that a peak on and a dark soliton coexist for the same wave speed. They have also discussed a loop solution for a special case that has been obtained by using numerical simulation.

In the present article, we focus our work on eq. (3) which describes unidirectional motion in shallow water. Where $u(x, t)$ is the fluid velocity and b represents the ratio of stretching to convection s.t., $b\neq0$ and $b\neq1$. Where $u\,u_x-\alpha^2\,u\,u_{xx}$ are the convection terms and $b\,u\,u_x-b\,\alpha^2\,u\,u_{xx}$ are the stretching terms.[1] Water is considered as the working fluid with height h = 5 cm over a flat bottom.

METHODS OF SOLUTION

In this work, we will study different solutions obtained using the factorization technique, the Cole-Hopf transformation method, and the Schwarzian derivative method, at the end we will study the stability of the system via the Phase Plane method.

The Factorization Technique

Recently different factorizations are used to solve the CH family equations. A factorization method is used to obtain the cusped soliton of the CH equation in parametric form Parker[16] without using the traveling wave transformation. Another factorization form used in Liu *et al*.[17] to solve the modified Camassa-Holm and Degasperis-Procesi equations and some new exact solitary wave solutions for the two equations are proposed. The figures for the bell-type and peakon-type solutions of the modified Camassa-Holm are plotted to describe the properties of the solutions. Under the traveling wave transformation, Fornberg-Whitham equation is reduced to an ordinary differential equation (ODE) whose general solution was obtained using the factorization technique.[18]

In the following part, we solve the CH-DP equation using a different form of factorization,[19] which to the best of our knowledge, has not been used to solve any of the CH family equations. Now, we consider the traveling wave transformation as follows:

$$\xi = x - c\ t \tag{4}$$

Where the fluid velocity is $u\ (x,t) = u(\xi)$ and its derivatives are:

$$u' = u_\xi = du / d\xi,$$

$$u'' = u_{\xi\xi} = d^2u / d\xi^2,\ u''' = u_{\xi\xi\xi} = d^3u / d\xi^3 ....,$$

In addition to that, c is a constant velocity which can be estimated using the linear dispersion relationship, via the linearization of eq. (3). This can be done by neglecting the nonlinear terms and making the substitution $u \approx e^{i\xi}$; such that to obtain.

$$c = -\frac{c_0 + \gamma}{\alpha^2 + 1} \tag{5}$$

Now, rewriting eq. (3) after passing to the traveling wave variable eq. (4), to convert eq. (3) into a nonlinear ODE; hence.

$$\begin{aligned}&\left(\alpha^2(c-u)+\gamma\right) u''' - \alpha^2 b u' u''\\ &+\left((b+1)\ u - c - c_0\right) u' = 0\end{aligned} \tag{6}$$

Integrating eq. (6) once with respect to ξ

$$\begin{aligned}&\left(\alpha^2(c-u)+\gamma\right) u'' - \alpha^2 \frac{b-1}{2} u'^2\\ &+\left(\frac{b+1}{2}\ u - c - c_0\right) u + K = 0\end{aligned} \tag{7}$$

Where K is an arbitrary integration constant. Then letting, $u=w-\Delta^{\pm}$, where:

$\Delta^{\pm} = \dfrac{-(c+c_0) \pm \sqrt{(c+c_0)^2 + 2K(b+1)}}{b+1}$ is a constant displacement. Moreover, let $a_1=c-\Delta$, $a_2 = \dfrac{b-1}{2}$, $a_3 = \dfrac{b+1}{2}$, $a_4=\Delta\ (b+1)+(c+c_0)$. Hence we can rewrite eq. (7) as follows:

$$w'' - \frac{a_2}{a_1 - w} w'^2 + \frac{\left(a_3\ w - a_4\right)}{\alpha^2 (a_1 - w)}\ w = 0 \tag{8}$$

Equation (8) is of the following type,

$$w'' + g(w)\ w'^2 + h\ (w)\ w' + k\ (w) = 0 \tag{9}$$

Factorizing eq. (9) in the non-commutative form proposed in Cornejo-Perez[19] as,

$$[D_\xi - \phi_1(w)\ w' - \phi_2(w)][D_\xi - \phi_3(w)]\ w = 0 \tag{10}$$

Under the conditions,

$$g(w) = -\varphi_1(w),$$

$$h\ (w) = \phi_1(w)\ \phi_3(w)\ w - \phi_2(w) - \phi_3(w) - \frac{\partial \phi_2(w)}{\partial w} w$$

$$k\ (w) = \phi_2(w)\ \phi_3(w)\ w \tag{11}$$

Eq. (8) can be factorized into the form (10) with the same conditions in (11). Then we have,

$$\phi_1(w) = \frac{a_2}{a_1 - w},\ \phi_2(w)\phi_3(w) = \frac{\left(a_3\ w - a_4\right)}{\alpha^2 (a_1 - w)} \tag{12}$$

$$\phi_1(w)\ \phi_3(w)\ w - \phi_2(w) - \phi_3(w) - \frac{\partial \phi_2(w)}{\partial w} w = 0 \tag{13}$$

Let $\varphi_2(w) = \dfrac{1}{a_5 \alpha^2 (a_1 - w)}, \varphi_3(w) = a_5(a_3 w - a_4),$

where a_5 is a fitting constant. Using simple algebra one can find, from eq. (13), that $a_1=c_0/2$, $a_2=-2$, $a_3=-1$, $a_4=c_0$, $b=-3$, $a_5 = \sqrt{-2/c_0^2\ \alpha^2}$ and $K = -\dfrac{c_0^2\left(\alpha^2 - 1\right) + 2c_0\gamma}{4\left(\alpha^2 + 1\right)}$. Therefore, the solution of $[D_\xi-\phi_3(w)]w=0$ will be in turn a solution of the factorized equation as well. The previous equation has the explicit form.

$$\frac{dw}{d\xi} - a_5(a_3 w - a_4)w = 0$$

By integrating, this leads to a permissible solution of eq. (3) taking the form,

$$u=\frac{a_4}{a_3+Exp\left[a_4a_5\left(\xi-\xi_0\right)\right]}-\Delta \quad (14)$$

We propose the initial-boundary condition $u=u_0=\beta_1\sqrt{g\ h_0}$ at ξ=0 where β_1 is a new dispersion parameter, similar to α, such that $|\beta_1|<1$. The parameter β_1 is chosen to satisfy the physical meaning. Then we have,

$$\beta_1\ \sqrt{g\ h_0}-\frac{a_4}{a_3+Exp\left[-a_4\ a_5\ \xi_0\right]}+\Delta=0 \quad (15)$$

Solving the previous equation leads to,

$$\xi_0=\frac{2\ i\ \pi\ C_4+\log\left[\frac{\Delta-\sqrt{g\ h_0}\beta_1}{a_3\sqrt{g\ h_0}\beta_1-a_4-a_3\Delta}\right]}{a_4\ a_5} \quad (16)$$

Substituting eqs. (14, 16) into eq. (6) at ξ=(x=0, t=0)=0, then after some algebraic manipulation we can find fourteen roots of γ as follows γ_1=0.644, $\gamma_{2,3}$=–0.481±0.362i, γ_4=–0.13908, γ_5=–0.13898, $\gamma_{6,7}$=–0.119±0.477i, γ_8=–0.031, γ_9=–0.0002, γ_{10}=0.133, $\gamma_{11,12}$=–0.181±0.308i, γ_{13}=0.268 and γ_{13}=15.938.

The Cole-Hopf Transformation

We introduce the Cole–Hopf transformation as follows Debnath,[20] it is also called the Backlund tranformation:

$$u(\xi)=A\frac{d^2}{d\xi^2}\log\varphi(\xi) \quad (17)$$

Where $\varphi'=\frac{d\varphi}{d\xi}$. This method can be applied to the CH-DP equation, the main steps read briefly as follows:

Step 1. By substituting Eq. (17) into the nonlinear ODE (6).

Step 2. With the aid of a symbolic software, equating the coefficients of different powers of $\varphi(\xi)^{-n}$ to zero, n={1,2,3,4,5,6,7}. We get seven equations.

Step 3. By solving those equations; we have the following results:

From the coefficient corresponding to $\varphi(\xi)^{-7}$ we have the following equation:

$$\left(2+b\right)A\alpha^2\left(\varphi'\left(\xi\right)\right)^7=0 \quad (18)$$

Taking φ'=0 will lead to a trivial solution while A and α^2 do not equal zero, as a result we will consider b=–2, as a solution of the eq. (18).

From the coefficient corresponding to $\varphi(\xi)^{-6}$, we have the following equation:

$$-42\ \left(b+2\right)\ A\alpha^2\ \left(\varphi'\left(\xi\right)\right)^5\varphi''\left(\xi\right)=0 \quad (19)$$

Which is satisfied for b=–2. Then, from the coefficient corresponding to $\varphi(\xi)^{-1}$, we have the following equation:

$$-\ \left(c+c_0\right)\ \varphi\left(\xi\right)+\left(c\alpha^2+\gamma\right)\ \varphi''\left(\xi\right)=0 \quad (20)$$

Which can be solved to determine a solution of $\varphi(\xi)$ as follows:

$$\varphi(\xi)=C_1Exp\left[-\frac{\sqrt{c+c_0}}{\sqrt{c\alpha^2+\gamma}}\xi\right]+C_2Exp\left[\frac{\sqrt{c+c_0}}{\sqrt{c\alpha^2+\gamma}}\xi\right] \quad (21)$$

We use the initial-boundary condition proposed earlier $u_0=\beta_2\sqrt{g\ h_0}$ at x=t=0 i.e., ξ=0 where β_2 is a new dispersion parameter, similar to α, such that $|\beta_2|<1$. Then we have an equation, after that substituting the value of φ into the equations of the coefficients of $\varphi(\xi)^{-5}$, $\varphi(\xi)^{-4}$, $\varphi(\xi)^{-3}$ and $\varphi(\xi)^{-2}$ at ξ=0 and b=–2 therefore we have the following set of equations, letting that $C_3=C_2/C_1$ and $\lambda=\sqrt{c+c_0}\big/\sqrt{c\alpha^2+\gamma}$,

$$A\ \sqrt{g\ h_0}\left(1+C_3\right)^2-4\ \beta_2\ C_3\ \lambda^2=0, \quad (22)$$

$$\left(C_3-1\right)^2\left(A+12\left(c\ \alpha^2+\gamma\right)\right)\lambda^2-A\ \alpha^2\lambda^4\left(C_3^2-14\ C_3+1\right)=0, \quad (23)$$

$$60\left(C_3-1\right)^2\left(c\ \alpha^2+\gamma\right)+A\left(5C_3^2\left(1-\ \alpha^2\lambda^2\right)+2C_3\left(29\alpha^2\lambda^2-5\right)-5\left(\alpha^2\lambda^2-1\right)\right)=0, \quad (24)$$

$$\lambda^2\left(A+15\left(c\ \alpha^2+\gamma\right)\right)-A\alpha^2\lambda^4-3\left(c-c_0\right)=0, \quad (25)$$

$$\left(c+c_0\right)\left(C_3-1\right)^2-\lambda^2 2A\left(C_3^2+C_3+1\right)$$
$$+5\left(5C_3^2+2C_3+5\right)\left(c\,\alpha^2+\gamma\right)$$
$$+2A\alpha^2\lambda^4\left(C_3^2-5C_3+1\right)=0 \tag{26}$$

Then solving simultaneously Eqs. (22, 23, 25, 26) will lead to the following four possibilities,

$$A=-\frac{(0.92\mp 0.34\,i)\,g\,h_0\beta}{\sqrt{g\,h_0}},\ \alpha=-3.56\times10^{-9}$$
$$\pm 2.22\times10^{-10}i,\ C_3=1.41\mp1.72\,i,$$
$$\gamma=\frac{2\times10^{-9}}{\sqrt{g\,h_0}}(7.45\times10^{-9}\ c_0\sqrt{g\,h_0}+$$
$$\left(3.86\times10^7\mp1.44\times10^7 i\right)\,g\,h_0\beta_2). \tag{27}$$

Finally, solving simultaneously Eqs. (22, 24, 25, 26) will lead to,

$$A=\frac{0.29\,g\,h_0\beta_2}{\sqrt{g\,h_0}},\ \alpha=\pm1.59\times10^{-9}i,\ C_3=-2.82,$$

$$\gamma=\frac{2.75\times10^{-11}}{\sqrt{g\,h_0}}(-4.77\times10^{-7}c_0\sqrt{g\,h_0}-8.92$$
$$\times10^8 g\,h_0\beta_2) \tag{28}$$

Furthermore, the substitution of eq. (21) into eq. (17) gives a solution of u as follows:

$$u(\xi)=A\left(\frac{\lambda^2\exp(\lambda\,\xi)+C_3\,\lambda^2\exp(-\lambda\,\xi)}{\exp(\lambda\,\xi)+C_3\exp(-\lambda\,\xi)}-\frac{\lambda^2\exp(\lambda\,\xi)+C_3\,\lambda^2\exp(-\lambda\,\xi))^2}{\exp(\lambda\,\xi)+C_3\exp(-\lambda\,\xi))^2}\right) \tag{29}$$

The Schwarzian Derivative Method

We introduce the Schwarzian derivatives transformation as follows Ovsienko and Tabachnikov[21]

$$u(\xi)=\frac{\psi'''(\xi)}{\psi'(\xi)}-\frac{3}{2}\left(\frac{\psi''(\xi)}{\psi'(\xi)}\right)^2 \tag{30}$$

Where $\psi'(\xi)=d\psi'(\xi)/d\xi$. This method can be applied to the CH-DP equation, the main steps read briefly as follows;

Step 1. By substituting eq. (30) into the nonlinear ODE eq. (6).

Step 2. With the aid of a symbolic software, equating the coefficients of $(\psi'(\xi))^{-j}$ to zero, $j=\{1,2,3,4,5,6,7\}$ i.e. we get seven equations.

Step 3. Solving those equations lead to the following results;

From the coefficient of $(\psi'(\xi))^{-7}$ we have,

$$27\,(b+2)\,\alpha^2\,\psi''(\xi)^7=0 \tag{31}$$

Since α, $\psi''(\xi)\neq0$ so $b=-2$. Then from the coefficient corresponding to $(\psi'(\xi))^{-6}$ we have the following equation:

$$-\frac{3}{2}(58b+111)\alpha^2\psi''(\xi)^5\psi'''(\xi)=0 \tag{32}$$

Solving the previous equation leads to

$$\psi(\xi)=K_2\,\xi^2+K_3\,\xi+K_4 \tag{33}$$

Where K_2, K_3 and K_4 are arbitrary integration constants. Substituting (33) into (30) to obtain a solution of u as follows:

$$u(\xi)=-\frac{6}{\left(K_5+2\,\xi\right)^2} \tag{34}$$

Where K_5 is an arbitrary constant s.t., $K_5=K_3/K_2$. Using the initial-boundary condition, which was mentioned earlier, K_5 can be determined s.t.,

$$K_5=\pm\frac{\mathrm{i}\sqrt{6}}{\sqrt{\beta_2}\left(g\,h_0\right)^{\frac{1}{4}}} \tag{35}$$

Substituting (33) into the equation of the coefficient corresponding to $(\psi'(\xi))^{-5}$ leads to,

$$-36\,K_2\,\left(1+b-8\,c\,\alpha^2-8\,\gamma\right)=0$$

Solving the previous equations with the dispersion relation (5) leads to the following:

$$c=-\frac{1}{8}\left(8\,c_0+b+1\right) \tag{36}$$

$$\gamma=\frac{1}{8}\left(\alpha^2\left(8\,c_0+b+1\right)+b+1\right). \tag{37}$$

Finally substituting eq. (33) into the coefficients corresponding to $(\psi'(\xi))^{-n}$, $n = \{1,2,3,4\}$ will satisfy each of these equations to be equal to zero. From the coefficient corresponding to $(\psi'(\xi))^{-4}$ we have:

$$-\frac{1}{2}\psi''(\xi)\begin{pmatrix} 4\ (8b+21)\ \alpha^2\ \psi'''(\xi)^3 + \\ (74\ b+93)\alpha^2\psi''(\xi)\psi'''(\xi)\psi^{\text{iv}}(\xi) \end{pmatrix}$$

$$-6\psi''(\xi)^2\begin{pmatrix} \left(3b-29\alpha^2 c-29\gamma+3\right)\psi''' \\ (\xi)-(b+3)\alpha^2\psi^{\text{v}}(\xi) \end{pmatrix} = 0 \quad (38)$$

From the coefficient corresponding to $(\psi'(\xi))^{-3}$ we have:

$$2\ (4b+13)\ \alpha^2\ \psi^{\text{iv}}(\xi)\ \psi'''(\xi)^2 - 3\ \psi''(\xi)^2$$
$$\left((b-18\ \alpha^2\ c-18\ \gamma+1)\ \psi^{\text{iv}}(\xi)-\alpha^2\ \psi^{\text{vi}}(\xi)\right)$$
$$+2\ \psi''(\xi)$$
$$\left(5\ \alpha^2\ b\ \psi^{\text{iv}}(\xi)+2\ (2b+3)\ \alpha^2\ \psi^{\text{v}}(\xi)\ \psi'''(\xi)\right.$$
$$\left.-6\ (c+c_0)\ \psi''(\xi)^3+\begin{pmatrix} 42\ (\alpha^2 c+\gamma) \\ -4\ (b+1) \end{pmatrix}\psi'''(\xi)^2\right) = 0 \quad (39)$$

From the coefficient corresponding to $(\psi'(\xi))^{-2}$ we have

$$\left(1+b-13\left(\alpha^2 c+\gamma\right)\psi^{\text{iv}}(\xi)-\alpha^2\psi^{\text{vi}}(\xi)\right)\psi'''(\xi)$$
$$-b\alpha^2\psi^{\text{iv}}(\xi)\psi^{\text{v}}(\xi)$$
$$+\left(4(c+c_0)\psi'''(\xi)-6\left(\alpha^2 c+\gamma\right)\psi^{\text{v}}(\xi)\right)\psi''(\xi)=0 \quad (40)$$

From the coefficient corresponding to $(\psi'(\xi))^{-1}$ we have

$$\left(\alpha^2 c+\gamma\right)\ \psi^{\text{vi}}(\xi)-(c+c_0)\ \psi^{\text{iv}}(\xi)=0 \quad (41)$$

DISCUSSION AND CONCLUSION

In this work we solved the CH-DP equation via three different methods, the factorization technique, the Cole-Hopf transformation and the Schwarzian derivative method. Through these methods of solution we determined two values of b which are $b=-3$ by applying the factorizations technique, and $b=-2$ resorting to both of the Cole-Hopf transformation and the Schwarzian derivative method.

In Elboree[22] the conservation laws were constructed in addition to the corresponding conserved quantities for the modified Camassa-Holm equation and (2+1) dimensional Zakharov-Kuznetsov -Benjamin-Bona-Mahoney equation, in this work we have made a preliminary test of conservation of our solutions eqs. (14, 29, 34) such that, for the solution by factorization, eq. (14), at γ_1 and γ_5 respectively give;

$$\lim_{\substack{x\to\pm\infty \\ t\to\infty}} u \approx -0.049,\ \lim_{\substack{x\to\pm\infty \\ t\to\infty}} u \approx -0.017 \quad (42)$$

Whereas the solution by the Cole-Hopf transformationthrough both conditions of eq. (27) and of eq. (28) show that:

$$\lim_{\substack{x\to\pm\infty \\ t\to\infty}} u \approx 0 \quad (43)$$

Finally, the solution obtained by using the Schwarzian derivative method is approaching zero as $\xi\to\alpha$ in the denominator of $u(\xi) = -\frac{6}{\left(K_5+2\ \xi\right)^2}$, i.e.

$$\lim_{\substack{x\to\pm\infty \\ t\to\infty}} u \approx 0 \quad (44)$$

Which shows that the momentum and kinetic energy are conserved for each of the obtained solutions:

$$-\infty < \lim_{\substack{x\to\pm\infty \\ t\to\infty}} (mu\ \ ,\frac{1}{2}m\ u^2) < \infty \quad (45)$$

First, in the case of the factorizations technique, we choose, without loss of generality, β_1 =0.9. For the first, fourth, fifth, eighth, ninth, tenth, and thirteenth roots of γ we obtain a kink solution for the real part and a dark Sech-like solitonssolution for the imaginary part at γ_1 [Figure 1]while we have a bright Sech-like solitons solution for the imaginary part at γ_5 [Figure 2]. Finally, the velocity is in the range of cm/s.

Second, we obtained different solitary wave patterns in the case of Cole-Hopf transformation. In [Figure 3] we see an exchange between ramps and dark solitons in the real part. While in the imaginary part of this solution we see an exchange

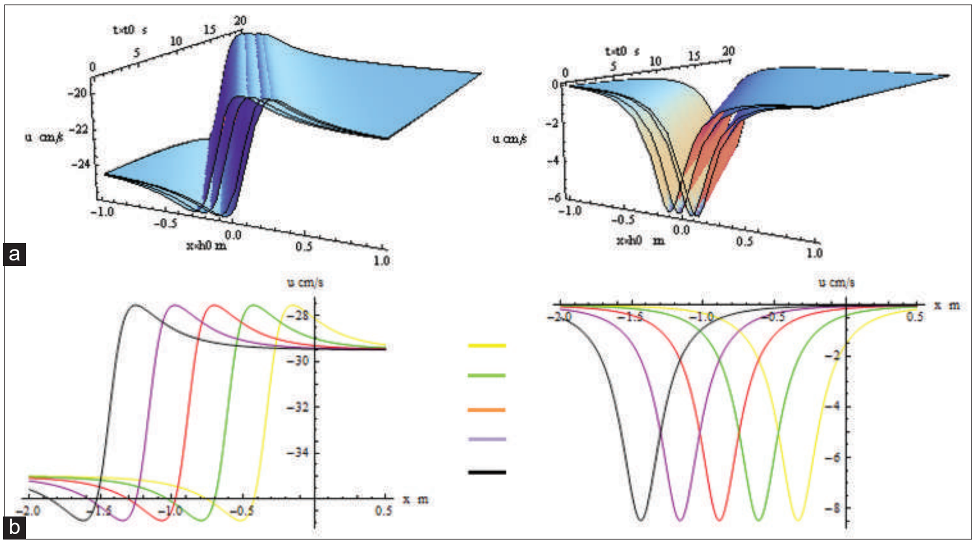

Figure 1: (a) The factorizations solution at $\gamma_{1,}$ β=0.1, 0.3, 0.5, 0.7, 0.9. (b) The factorizations solution at $\gamma_{1,}$ t=5, 10, 15, 20, 25

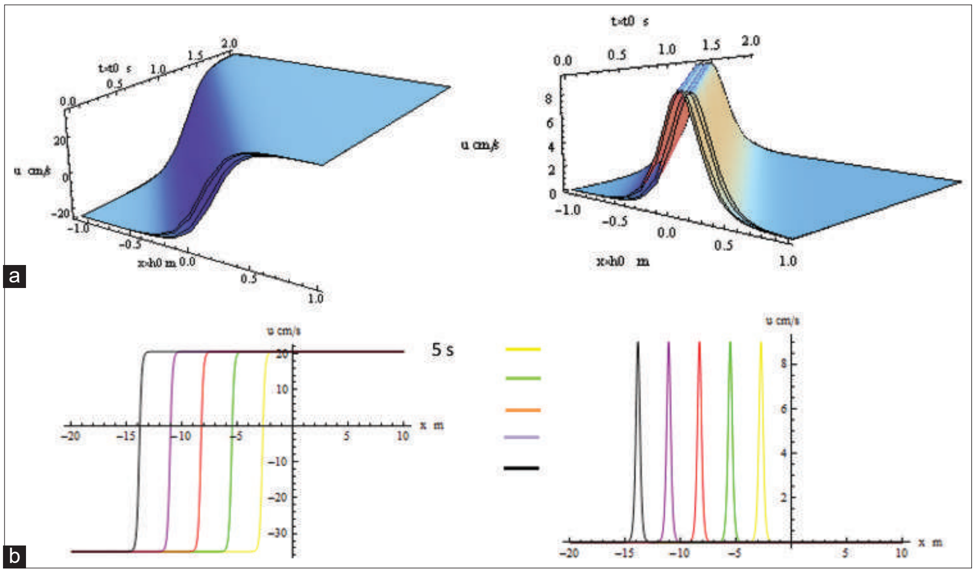

Figure 2: (a) The factorizations solution at $\gamma_{5,}$ β=0.7, 0.75, 0.8, 0.85, 0.9. (b) The factorizations solution at $\gamma_{5,}$ t=5, 10, 15, 20, 25

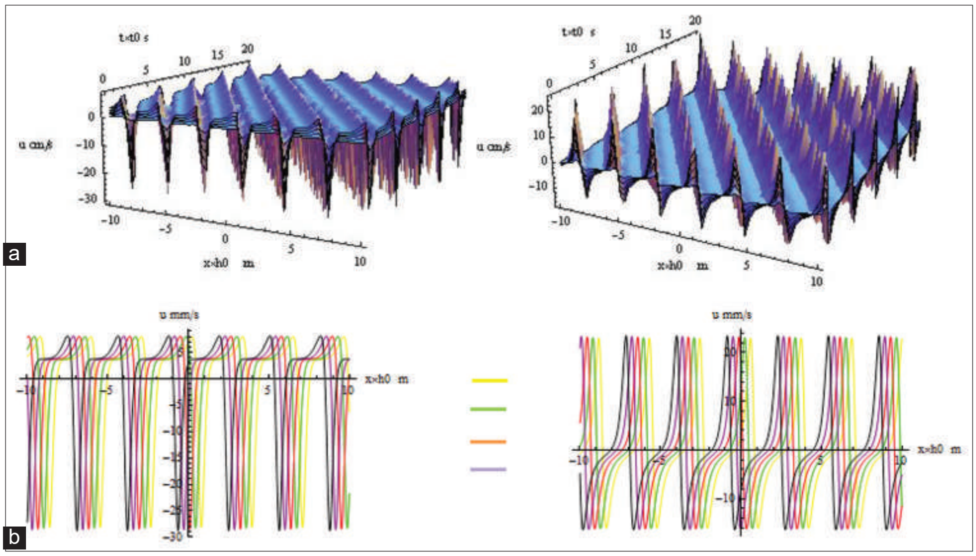

Figure 3: (a) The Cole-Hopf transformation solution at the condition of eq. (27) (real and imaginary parts respectively) at β=0.01, 0.03, 0.05, 0.07, 0.09. (b) The Cole-Hopf transformation solution at the conditions of eq. (27) (real and imaginary parts respectively) at t=5, 10, 15, 20, 25

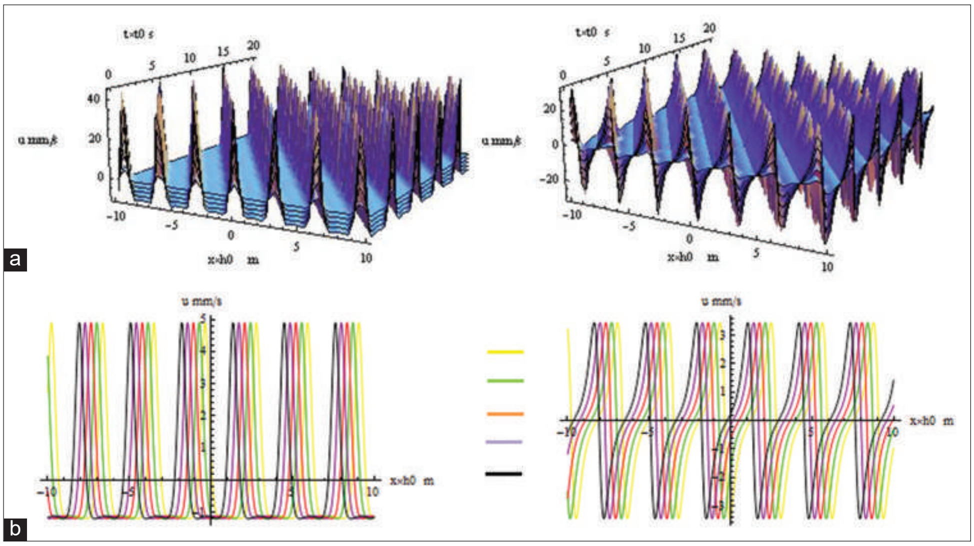

Figure 4: (a) The Cole-Hopf transformation solution at the condition of eq. (28) (real and imaginary parts respectively) at β=0.01, 0.03, 0.05, 0.07, 0.09. (b) The Cole-Hopf transformation solution at the condition of eq. (28) (real and imaginary parts respectively) at t=5, 10, 15, 20, 25

between ramps, bright solitons, and cliffs. In [Figure 4] we see a soliton solution in the real part, while we see an exchange of ramps, solitons, and cliffs in the imaginary part of the solution. Finally, using negative value of β leads to a rotation of 180°.

Ramps and cliffs pattern was explained in details in Holm and Staley[1] [Figure 5] as it was one of the observed patterns together with the peakons and leftons. In addition, they had exchange between ramps and peakons [Figure 6] or ramps and leftons.

Finally, illustrating the solution obtained via the Schwarzian derivative method, in [Figure 7] we see a bright Sech-like soliton for the real part while we have an exchange between dark and bright Sech-like solitons for the imaginary part.

In Holm and Staley[1] some traveling wave solutions were obtained at γ=0 for different values of b. The numerical results obtained there are peakons for 1<b, ramps, and cliffs for $-1<b<1$ and leftons for $b<-1$ in the velocity profile emerges under the peakon eq. (3) for a set of Gaussian initial conditions for different values of α. In addition, a nearly stationary pattern was illustrated at $b=-1$.

In this work a variety of wave patterns result when illustrating the different solutions we obtained. The kink pattern and dark sech-like pattern when illustrating the solution obtained using the Factorizations technique for $b=-3$ [Figures 1 and 2]. Sech like pattern is also known as smooth solitary wave.[23] Similar pattern formation

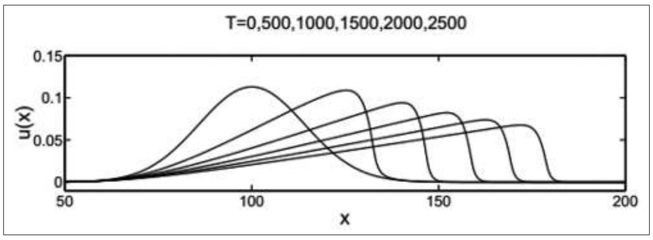

Figure 5: Ramps and cliffs pattern

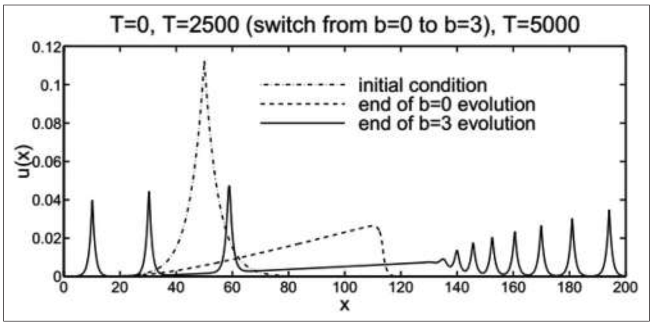

Figure 6: Exchange between ramps and peakons

dark sech-like is noticed in Laiq and Rashid, Yıldırım[24,25] for the solution of the modified Camassa-Holm (at b=2) and the modified Degasperis-Procesi (at b=3) equations. While the solitons, ramps and cliffs patterns were shown when illustrating the solutions obtained using the Cole-Hopf transformation [Figures 3 and 4], Finally, the Schwarzian derivatives method Figure 7 where $b=-2$ show bright and dark sech-like solitons. The difference between peakons and solitons is that peakons are solitons with discontinuous derivtivates at the peak, so studying the derivatives of our solutions we found that their derivatives are continuous.In addition to that, we

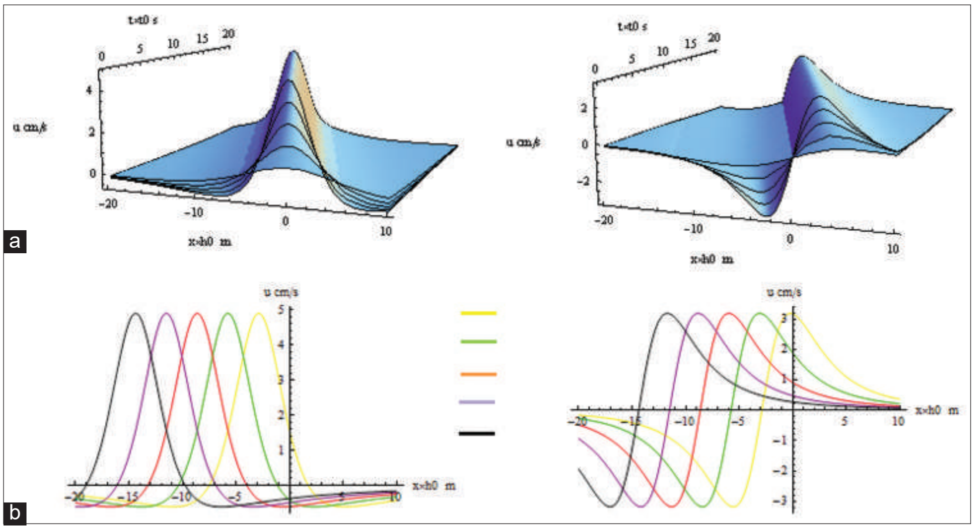

Figure 7: (a) The Schwarzian derivatives solution (real and imaginary parts respectively) at β=0.01, 0.02, 0.04, 0.06, 0.08. 0.1, (b) The Schwarzian derivatives solution (real and imaginary parts respectively) at t=5, 10, 15, 20, 25

have found that the proposed dispersion parameter β is in direct proportion with the velocity since increasing the value of β slightly increases the velocity u (x,t). In addition to that, we noticed a displacement over the x axis to the whole pattern when increasing the value of β i.e., interference of the waves without changing the pattern.

STABILITY OF THE SYSTEM VIA THE PHASE PORTRAIT METHOD[26,27]

Returning back to the variable w in eq. (8), let that

$$w' = y = f_1(w, y) \tag{46}$$

$$w'' = y' = \frac{a_2\alpha^2 y^2 - a_3 w^2 + a_4 w}{a_1\alpha^2 - w} = f_2(w, y) \tag{47}$$

In order to determine the critical points we set f_1 (w, y) = 0, leading to y = 0, and f_2 (w, y) = 0, which leads to $a_2\, a^2\, y^2 - a_3\, w^2 + a_4\, w=0$. Then at y=0, $-a_3\, w^2 + a_4\, w=0$ leading to $w_1=a_3/a_4$ or $w_2=0$. Then, using the numerical values of the parameters which were used in the case of factorization we can find that $a_3/a_4=0.05$ at γ_1 while $a_4/a_3=0.55$ at γ_5. So, the equilibrium points are (0, 0) and (0.05, 0) γ_1. While the equilibrium points at γ_5 are (0, 0) and (0.55, 0). Now, to determine the nature of the critical points, we need to find the eigenvalues for each one of them. First we construct the following Jacobian matrix.

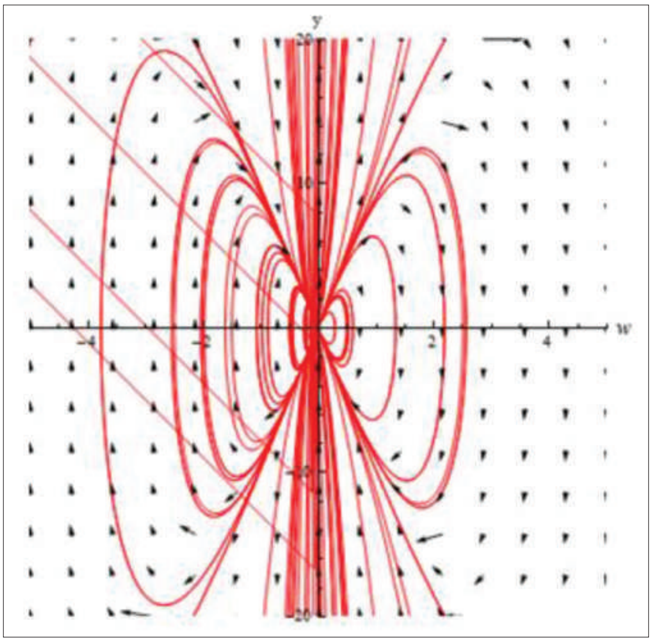

Figure 8: "Phase portrait" of the velocity u, at γ_1 stable saddle at (0, 0) and a center at (0.05,0)

$$J = \begin{pmatrix} \partial f_1/\partial w & \partial f_1/\partial y \\ \partial f_2/\partial w & \partial f_2/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \dfrac{a_3 w^2 + a_1(a_4 - 2a_3 w) + a_2\alpha^2 y^2}{\alpha^2 (a_1 - w)^2} & \dfrac{2a_2 y}{a_1 - w} \end{pmatrix}$$

The eigenequation $|J - \eta\, I| = 0$ gives the eigenvalues η, where I is the identity matrix.

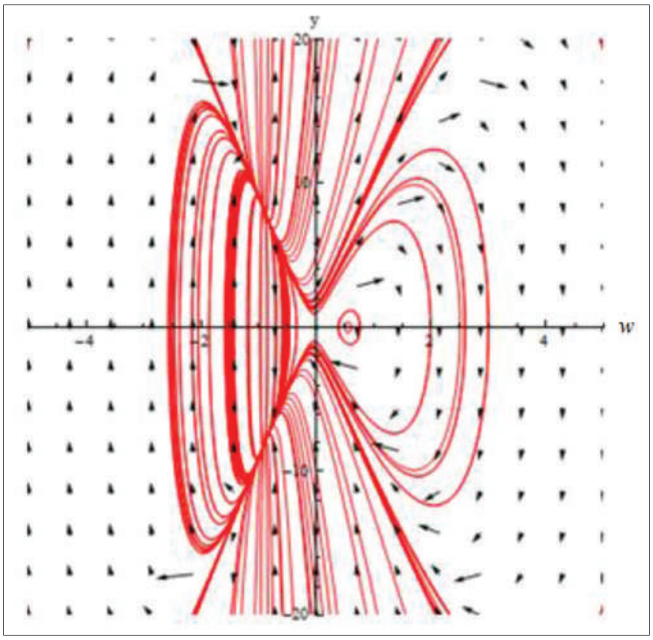

Figure 9: "Phase portrait" of the velocity u, at γ_5 stable saddle at (0, 0) and a center at (0.55,0)

$$|J-\eta\, I| =$$

$$\begin{vmatrix} 0-\eta & 1 \\ \dfrac{a_3 w^2 + a_1\left(a_4 - 2a_3 w\right) + a_2\alpha^2 y^2}{\alpha^2\left(a_1 - w\right)^2} & \dfrac{2a_2 y}{a_1 - w} - \eta \end{vmatrix}$$

This shows that we have, at γ_1, a stable saddle point at (0, 0) because $\eta_{1,2}$=±13.6 and a center at (0.05, 0) as $\eta_{3,4}$=±3.5i. Then, at γ_5 we have also a stable saddle point at (0, 0) because $\eta_{5,6} = \pm 61.3$ and a center at (0.55,0) as $\eta_{7,8}$=±6.2i. By the aid of symbolic software, the phase portrait is illustrated, using the values of the parameter used in the case of factorization, in Figures 8 and 9.

REFERENCES

1. Holm DD, Staley MF. Wave structure and nonlinear balances in a family of evolutionary PDEs. Siam J Appl Dyn Syst 2003;2:323-80.
2. Tian L, Fan J, Tian R. The attractor on viscosity Peakon b-family of equations. Int J Nonlinear Sci 2007;4:163-70.
3. Camassa R, Holm DD. An integrable shallow water equation with peaked solitons. arXiv 1993;13:9305002.
4. Degasperis A, Procesi M, Asymptotic integrability. In: Degasperis A, Aeta G, editors. Symmetry and Perturbation Theory. Singapore: World Scientific; 1999. p. 23-7.
5. Wang CY, Guan J, Wang BY. The classification of single travelling wave solutions to the Camassa-Holm-Degasperis-Procesi equation for some values of the convective parameter. Pramana J Phys 2011;77: 759-64.
6. Constantin A, Lannes D. The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equation. Arch Rat Mech Anal 2009;192:165-86.
7. Mhlanga IE, Khalique CM. Solutions of two nonlinear evolution equations using lie symmetry and simplest equation methods. Mediterr J Math 2014;11:487-96.
8. Lenells J. Classification of all travelling-wave solutions for some nonlinear dispersive equations. Philos Trans R Soc A 2007;365:2291-8.
9. Ivanov R. Conformal and geometric properties of the Camassa-Holm hierarchy. arXiv 2009;2009:1107.
10. Nuseir AS. A direct method for solving nonlinear PDEs and new exact solutions for some examples. Int J Contemp Math Sci 2011;6:2283-90.
11. Xie S, Lin Q, Gao B. Periodic and solitary travelling-wave solutions of a CH-DP equation. Commun Nonlinear Sci Numer Simulat 2011;16:3941-8.
12. Xie S, Wang L. Compacton and generalized kink wave solutions of the CH-DP equation. Appl Math Comput 2010;215:4028-39.
13. Li J, Zhang Y. Exact loop solutions, cusp solutions, solitary wave solutions and periodic wave solutions for the special CH-DP equation. Nonlinear Anal 2009;10:2502-7.
14. Fan X, Yang S, Yin J, Tian L. Peakon, loop and solitary travelling wave solutions for the general modified CH DP equation. Appl Math Comput 2011;217:4812-8.
15. Xie S, Du R. Periodic loop solutions of the CH-DP equation. Indian J Sci Technol 2010;3:227-30.
16. Parker A. Cusped solitons of the Camassa-Holm equation. I. Cuspon solitary wave and antipeakon limit. Chaos Solitons Fractals 2007;34:730-9.
17. Liu Y, Zhu X, He J. Factorization technique and new exact solutions for the modified Camassa-Holm and Degasperis-Procesi equations. Appl Math Comput 2010;217:1658-65.
18. Feng C, Wu C. The classification of all single traveling wave solutions to Fornberg-Whitham equation. Int J Nonlinear Sci 2009;7:353-9.
19. Cornejo-Perez O. Exact solutions of a flat full causal bulk viscous FRW cosmological model through factorization. arXiv 2012;2012:5938.
20. Debnath L. Nonlinear Partial Differential Equations for Scientists and Engineers. Boston, USA: Birkhäuser; 1997.
21. Ovsienko V, Tabachnikov S. What is the Schwarzian derivative? AMS Notices 2009;56:34-6.
22. Elboree MK. Conservation laws, soliton solutions for modified Camassa-Holm equation and (2+1)-dimensional ZK-BBM equation. Nonlinear Dyn 2017;89:2979-94.
23. Rui W, He B, Xie S, Long Y. Application of the integral bifurcation method for solving modified Camassa-Holm and Degasperis-Procesi equations. Nonlinear Anal 2009;71:3459-70.
24. Laiq ZA, Rashid NA. An efficient approach for solution of modified Camassa-Holmand Degasperis-Procesi equations. U P B Sci Bull Series D 2017;79:1-10.
25. Yıldırım A. Variational iteration method for modified Camassa-Holm and Degasperis-Procesi equations. Int J

Numer Meth Biomed Eng 2010;26:266-72.
26. Niu X. Transformations from the unperturbed equation to the perturbed equation with nth order perturbation. Commun Nonlinear Sci Numer Simulat 2008;13:1241-5.
27. Jordan D, Smith P. Nonlinear ordinary differential equations: An introduction for scientists and engineers. In: Oxford Texts in Applied and Engineering Mathematics. Oxford, United Kingdom: OUP Oxford; 2007.