AJMS
Asian Journal of Mathematical Sciences

RESEARCH ARTICLE

6TH-ORDER RUNGE-KUTTA FORWARD-BACKWARD SWEEP ALGORITHM FOR SOLVING OPTIMAL CONTROL MODELS OF EPIDEMIOLOGICAL TYPE

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Received: 11-10-2024; Revised: 19-11-2024; Accepted: 05-12-2024

ABSTRACT

This paper seeks to formulate a more accurate forward-backward algorithm for solving optimal control problems using the 7-stage Runge-Kutta of order 6 (RK6) numerical scheme. The control variable were approximated using the interpolating polynomial or spine while the RK6 forward and backward sweeps were used in approximating the state and adjoint variables respectively because its A-stability, accuracy and higher rate of convergence. Three numerical examples were simulated to ascertain the accuracy and convergence of the 6th order Runge-Kutta forward-backward sweep method (K6FBSM). It was discovered that the RK6FBSM performs better when compared with the Euler and the 4th order Runge-Kutta Forward-Backward Sweep method RK4FBSM.

Keywords: Runge-Kutta of order 6, Forward-Backward Sweep method,

INTRODUCTION

Optimal control involves finding control inputs that optimize a particular performance criterion subject to system dynamics and constraints[1]. The modern formulation of optimal control theory began in the mid-20th century with contributions from Plethora of authors[2], The Forward Backward Sweep Method (FBSM) is an indirect numerical method widely used for solving optimal control problems due to its computational efficiency and ease of implementation[3]. Reviewed the FBSM for both bounded and unbounded control problems, incorporating various numerical schemes such as Euler, trapezoidal, and Runge-Kutta techniques[4]. The method iteratively solves the state equations forward in time and the adjoint equations backward in time, updating the control inputs at each iteration[5]. The Forward-Backward Sweep (FBS) method is a popular technique for solving optimal control problems, especially those formulated as two-point boundary value problems by iteratively solving the state and costate equations forward and backward in time, respectively[6]. According to the FBS method is particularly effective because it directly integrates the state and costate equations while updating the control policy at each iteration[7]. The method is proven to be robust and provides accurate solutions for a range of test problems, demonstrating its utility in practical applications[8].

METHODOLOGY

The solution to optimal control problems involves the derivation of the optimal control characterization and the use of the forward and backward sweep for the state and control trajectories respectively[9]. By the Pontryagin's maximum, the optimal control can be analytically derived by the optimality condition while the state and control variables can be derived numerically using the Runge-Kutta of order six subject to the adjoint and transversality conditions[10].

1.1: Statement of problem

We considered the generalized Optimal Control problem given below;

$$\max(\min) J(u) = \int_{t_0}^{T} F(t, x_1(t), x_2(t), \ldots, x_n(t), u(t))\, dt \quad (1)$$

$$\text{subject to}: \dot{x}_1 = g_1(t, x_1(t), x_2(t), \ldots, x_n(t), u(t)) \quad (2)$$

$$\dot{x}_2 = g_2(t, x_1(t), x_2(t), \ldots, x_n(t), u(t)) \quad (3)$$

...

$$x_n = g_n(t, x_1(t), x_2(t), \ldots, x_n(t), u(t)) \quad (4)$$

$$u_{\min} \leq u \leq u_{\max} \quad (5)$$

where $(x_1(t_0), x_2(t_0), \ldots, x_n(t_0)) = (0,0,\ldots,0)^T \in \mathbf{R}^n$.

The Hamiltonian of the constrained optimal control problem is written as

$$H = F(t, x_1(t), x_2(t), \ldots, x_n(t), u(t)) + \sum_{i=1}^{n} \lambda_i g_i(t, x_1(t), x_2(t), \ldots, x_n(t), u(t)), \quad (6)$$

and the optimality, adjoint and transversality conditions are expressed below respectively as

(Optimality) $$\frac{\partial H}{\partial u} = \frac{\partial F}{\partial u} + \sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial u} \quad (7)$$

(Adjoint) $$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} = -\frac{\partial F}{\partial x_i} - \sum_{i=1}^{n} \lambda_i \frac{\partial g_i}{\partial x_i}, \quad (8)$$

(Transversality), $$\lambda_i(T) = 0 \;\; \forall\, i = 1, 2, \ldots, n \quad (9)$$

1.2: 6th Order Runge-Kutta Scheme

The 7-stage Runge-Kutta of order 6 (RK6) iterative scheme was considered for the development of the forward-backward sweep Algorithm for the solution of Optimal control problems[11]. The RK6 enhances the level of accuracy of the results of the optimal control problem with the developed algorithm[12].

$$K_1 = hf(t_k, x_k),$$
$$K_2 = hf\left(t_k + \frac{h}{4}, x_k + \frac{K_1}{4}\right),$$
$$K_3 = hf\left(t_k + \frac{h}{4}, x_k + \frac{K_1 + K_2}{8}\right),$$
$$K_4 = hf\left(t_k + \frac{h}{2}, x_k + \frac{-5K_2 + 8K_3}{6}\right),$$
$$K_5 = hf\left(t_k + \frac{3h}{4}, x_k + \frac{K_1 + K_2 + 4K_4}{8}\right),$$
$$K_6 = hf\left(t_k + \frac{3h}{4}, x_k + \frac{3K_2 + 2K_3 - K_4 + 2K_5}{8}\right),$$
$$K_7 = h\,f\left(t_k + h, x_k + \frac{K_1 - 2K_2 + 4K_3 + 4K_6}{7}\right)$$
$$x_{k+1} = x_k + \frac{7K_1 + 32K_3 + 12K_4 + 16K_5 + 16K_6 + 7K_7}{90}$$

1.3: Forward-Backward Sweep Method

The derivation of the forward-backward sweep requires the discretization of state, control, and adjoint variables along the knots $t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_N$, such that $x_i^k = x_i(t_0 + kh)$,

$u_s^k = u_s(t_0 + kh)$, and $\lambda_{ji} = \lambda_i(t_0 - jh)$ for $t_k = t_0 + kh$ and $h = (T - t_0)/N$ is the

Step-length with N number of grid-points.Therefore, for any argument x_i in the state vector ($x_i^k, x_2^k, \cdots, x_n^k) \in \mathbb{R}^n$ and a unit adjoint variable u_s for $s \in \{1,2,\cdots,m\}$ and the forward sweep of the state variable using the RK6 method is written as:

$$K_{i1} = hg_1(t_k, x_i^k, u_s(k)), \tag{10}$$
$$K_{i2} = hg_2(t_k + \frac{h}{4}, x_i^k + \frac{K_1}{4}, u_s(t_k + \frac{h}{4})), \tag{11}$$
$$K_{i3} = hg_3(t_k + \frac{h}{4}, x_i^k + \frac{1}{8}(K_1 + K_2),\ u_s(t_k + \frac{h}{4})), \tag{12}$$
$$K_{i4} = hg_4(t_k + \frac{h}{2}, x_i^k + \frac{1}{6}(-5K_2 + 8K_3), u_s(t_k + \frac{h}{2})), \tag{13}$$
$$K_{i5} = hg_5(t_k + \frac{3h}{4}, x_i^k + \frac{1}{8}(K_1 + K_2 + 4K_3), u_s(t_k + \frac{3h}{4})), \tag{14}$$
$$K_{i6} = hg_6(t_k + \frac{3h}{4}, x_i^k + \frac{1}{8}(3K_2 + 2K_3 - K_4 + 2K_5), u_s(t_k + \frac{3h}{4})) \tag{15}$$
$$K_{i7} = h\,g_7\left(t_k + h, x_i^k + \frac{1}{7}(K_1 - 2K_2 + 4K_3 + 4K_6),\ u_s(t_k + h)\right) \tag{16}$$
$$x_i^{k+1} = x_i^k + \frac{7K_1 + 32K_3 + 12K_4 + 16K_5 + 16K_6 + 7K_7}{90}. \tag{17}$$

for the subscript of each argument $i = 1,2,\cdots,n$, $r = 1,2,\cdots,m$, the counter $k = 1,2,\cdots,N$. and the functions g_l, $l = 1, 2\cdots 7$ are once continuously differentiable within the time interval $[t_0, T]$ $\left(\text{i.e } \frac{\partial g_l}{\partial x_i} \in C^1[t_0, T]\right)$ and K_{il} denoting the dynamical function of the i−th component (argument) and the l−th stage. For the control characterization, the optimal variable is derived using the optimality condition $\frac{\partial H}{\partial u_r} = 0$ such that

$$u_r^* = \min\{\max\{u_{min}, u_r, \}, u_{max}\} \tag{18}$$

The interpolating polynomial or spline [8] of the control variable $u_s(.)$ can be computed as the formula;

$$u(t \underset{-}{+} \frac{ah}{b}) = \frac{au(t \underset{-}{+} h) + (b-a)u(t)}{b} \tag{19}$$

The control variable for the forward(+) and backward (-) sweep are approximated respectively as follows;

$$u(t \underset{-}{+} \frac{ah}{b}) \approx \begin{cases} \frac{u(t \underset{-}{+} h)+3u(t)}{4} & a=1, b=4 \\ \frac{u(t \underset{-}{+} h)+u(t)}{2} & a=1, b=2 \\ \frac{3u(t \underset{-}{+} h)+u(t)}{4} & a=3, b=4 \end{cases} \tag{20}$$

The backward sweep of the adjoint variables using the proposed RK6 method requires the discretization of the adjoint such that for any argument $\lambda_l^j \in (\lambda_1^j, \lambda_2^j, \cdots, \lambda_n^j) \in \mathbb{R}^n$ with a specific control variable u_s attached to each state equation is expressed thus:

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}(t, \lambda_i, x_i, u_s), \quad i = 1, 2, \ldots, n. \tag{21}$$

Deploying the 7-stage RK6 numerical scheme in the discretization of the backward sweep for the adjoint variables yields the following:

$$K_{i1} = -\frac{\partial H}{\partial x_i}(t_j, \lambda_i^j, x_i, u_s(j)), \tag{22}$$

$$K_{i2} = -\frac{\partial H}{\partial x_i}(t_j + \frac{h}{4}, \lambda_i^j + \frac{K_1}{4}, x_i^j + \frac{K_1}{4}, u_s(t_j + \frac{h}{4})), \tag{23}$$

$$K_{i3} = -\frac{\partial H}{\partial x_i}(t_j + \frac{h}{4}, \lambda_i^j + \frac{1}{8}(K_1 + K_2), x_i^j + \frac{1}{8}(K_1 + K_2), u_s(t_j + \frac{h}{4})), \tag{24}$$

$$K_{i4} = -\frac{\partial H}{\partial x_i}(t_j + \frac{h}{2}, \lambda_i^j + \frac{1}{6}(-5K_2 + 8K_3), x_i^j + \frac{1}{6}(-5K_2 + 8K_3), u_s(t_j + \frac{h}{2})), \tag{25}$$

$$K_{i5} = -\frac{\partial H}{\partial x_i}(t_j + \frac{3h}{4}, \lambda_i^j + \frac{1}{8}(K_1 + K_2 + 4K_3), x_i^j + \frac{1}{8}(K_1 + K_2 + 4K_3), u_s(t_j + \frac{3h}{4})), \tag{26}$$

$$K_{i6} = -\frac{\partial H}{\partial x_i}(t_j + \frac{3h}{4}, \lambda_i^j + \frac{1}{8}(3K_2 + 2K_3 - K_4 + 2K_5), x_i^j + \frac{1}{8}(3K_2 + 2K_3 - K_4 + 2K_5), u_s(t_j + 3h4)), \tag{27}$$

$$K_{i7} = -\frac{\partial H}{\partial x_i}(t_j + h, \lambda_i^j + \frac{1}{7}(K_1 - 2K_2 + 4K_3 + 4K_6), x_i^j + \frac{1}{7}(K_1 - 2K_2 + 4K_3 + 4K_6), u_s(t_j + h)), \tag{28}$$

$$\lambda_i^{j-1} = \lambda_i^j - \frac{h}{90}(7K_1 + 32K_3 + 12K_4 + 16K_5 + 16K_6 + 7K_7). \tag{29}$$

1.4: RK6 Forward-Backward Algorithm for Optimal Control Problem

Step 1: Initialization Input

$$x_i(t_0) = x_i^0, \quad \lambda_i(N) = 0 \; \forall i = 1, 2, \cdots, n, \; T, \; t_0, \; u_r(t_0) = u_r^0, \forall r = 1, 2, \cdots, m$$

Step 2: Forward Sweep for state variables

Compute while k = 0,1,2,...,N do x_i^{i+1} from equations (10) to (17) respectively and sequentially.

Step 3: Backward Sweep for adjoint variables

Set $j = N + 2 - i$ and compute λ_i^{j-1} from equations (22) to (29) respectively *Step 4: Control Characterization* Compute control within bounds $u_r^* = \min\{\max\{u_{min}, u_r,\}, u_{max}\}$ for $r = 1, \cdots, m$ from equation (18)

Step 5: Termination criteria

If termination conditions are met go to step 6 otherwise step 7

Step 6: Output $x^k{}_i, \lambda^j{}_i, u^*{}_i(\forall i,j)$ and end function *Step 7: Return* Repeat step 2

2: Numerical simulations: Implementation and Results

Example 1: Considering the optimal control problem below

$$\text{Min } J[u] = \min\int_0^1 u(t)2\, dt,$$

$$x'(t) = x(t) + u(t), \tag{30}$$

$$x(0) = 1,\ x(1) \text{ free.}$$

The Hamiltonian function H is defined as $H = u(t)^2 + \lambda(t)\cdot(x(t)+u(t))$ where $\lambda(t)$ is the adjoint variable (or costate). Using the optimality adjoint and transversality conditions yields the analytical (exact) optimal solution below:

$$x^*(t) = e^t, \qquad u^*(t) = 0. \tag{31}$$

The optimal control obtained using the optimality condition, $\frac{\partial H}{\partial u} = 0$, is given by

$$\begin{aligned} u^*(t) &= -\frac{\lambda(t)}{2}, \\ &= \min\left(u_{\max},\ \max\left(u_{min}, -\frac{\lambda(t)}{2}\right)\right) \end{aligned} \tag{32}$$

ascertained to be minimum since $\frac{\partial^2 H}{\partial u^2} = 2 > 0.$

The derived costate equation using the adjoint conditions, $\lambda'(t) = -\frac{\partial H}{\partial I(t)}$, given by:

$$\lambda'(t) = -\lambda(t), \quad \lambda(T) = 0 \tag{33}$$

Applying the forward Euler, RK4 and the proposed RK6 forward -backward sweep methods (i.e RK4FBSM and proposed RK6FBSM respectively) yields the results below.

Table1: Result of State variable for example 1

S/N	Exact	Euler		RK4FBSM		Proposed RK6FBSM	
	x_A	x_E	$\lvert x_A - x_E\rvert$	$xK4$	$\lvert xA - xK4\rvert$	$xK6$	$\lvert xA - xK6\rvert$

1	1.1051709181	1.1111111111	5.9401930000 $\times10^{-3}$	1.1051708333	8.47000 $\times10^{-8}$	1.1051709181	0.00000 $\times 10^{0}$
2	1.2214027582	1.2345679012	1.3165143100 $\times10^{-2}$	1.2214025709	1.87300 $\times10^{-7}$	1.2214027582	0.00000 $\times10^{0}$
3	1.3498588076	1.3717421125	2.1883304900 $\times10^{-2}$	1.3498584971	3.10500 $\times10^{-7}$	1.3498588076	0.00000 $\times10^{0}$
4	1.4918246976	1.5241579028	3.2333205100 $\times10^{-2}$	1.4918242401	4.57600 $\times10^{-7}$	1.4918246977	0.00000 $\times10^{0}$
5	1.6487212707	1.6935087808	4.4787510100 $\times10^{-2}$	1.6487206386	6.32100 $\times10^{-7}$	1.6487212707	0.00000 $\times10^{0}$
6	1.8221188004	1.8816764232	5.9557622800 $\times10^{-2}$	1.8221179621	8.38300 $\times10^{-7}$	1.8221188004	0.00000 $\times10^{0}$
7	2.0137527075	2.0907515813	7.6998873800 $\times10^{-2}$	2.0137516266	1.08090 $\times10^{-6}$	2.0137527075	0.00000 $\times10^{0}$
8	2.2255409285	2.3230573125	9.7516384100 $\times10^{-2}$	2.2255395633	1.36520 $\times10^{-6}$	2.2255409285	0.00000 $\times10^{0}$
9	2.4596031112	2.5811747917	1.2157168060 $\times10^{-1}$	2.4596014138	1.69740 $\times10^{-6}$	2.4596031112	0.00000 $\times10^{0}$
10	2.7182818285	2.8679719908	1.4969016230 $\times10^{-1}$	2.7182797441	2.08430 $\times10^{-6}$	2.7182818285	0.00000 $\times10^{0}$

Table 1: $|x_A - x_E|$, $|x_A - x_{K4}|$ and $|x_A - x_{K6}|$ ≡ errors of Euler RK4 & RK6 respectively

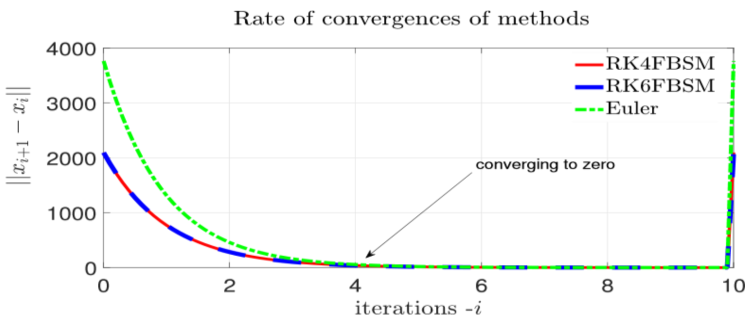


Figure 1: Rate of convergence in 10 iters.

Example 2: Considering the SIS Model with Treatment [9]

$$\min C[u] = \int_0^T \omega_1 I(t) + u^2(t)\big)\, dt,$$

$$s.t : \dot{I}(t) = \beta \; (N - I(t))I(t) - (\mu + \gamma)I(t) - u(t)I(t) \quad (34)$$

(35)

$$I(0) = I_0, I(T) free. \quad (36)$$

The Hamiltonian is given by:

$H(I(t),u(t),\lambda(t)) = \omega_1 I(t) + u^2(t) + \lambda(t)(\beta(N - I(t))I(t) - (\mu + \gamma)I(t) - u(t)I(t)$(37) The optimal control obtained using the optimality condition, $\frac{\partial H}{\partial u} = 0$, is given by

$$\begin{aligned} u^*(t) &= \frac{\lambda(t)I(t)}{2}, \\ &= \min\left(u_{\max}, \max\left(u_{min}, \frac{\lambda(t)I(t)}{2}\right)\right). \end{aligned} \quad (38)$$

ascertained to be minimum since $\frac{\partial^2 H}{\partial u^2} = 2 > 0.$

The derived co-state equation using the adjoint conditions, $\lambda'(t) = -\frac{\partial H}{\partial I(t)}$, given by:

$\lambda'(t) = -\omega_1 - \lambda(t)\beta(N - I(t)) - \beta I(t) - (\mu + \gamma) - u(t), \lambda(T) = 0$ (39)

Applying the forward Euler, RK4 and the proposed RK6 forward -backward sweep methods (i.e RK4FBSM and proposed RK6FBSM respectively) also yields the results in Table 2 below using the following parameters: $\beta = 0.05$, $\mu = 0.01$, $\gamma = 0.5$, $N = 100$, $\omega_1 = 1$ and $T = 1$.

Table 2: Result of State and Control variables for example 2

S/N	Euler		convergence		Proposed RK6FBSM	
	x_E	u_E	$xK4$	$uK4$	$xK6$	$uK6$
1	10.0000000000	5.2355651638	10.0000000000	4.5926947060	10.0000000000	4.6363573386
2	8.7650656568	4.6326137300	9.5613737942	4.3167361686	9.5464589222	4.3768363778
3	8.2556597311	4.1600567381	9.4089685710	4.0418362859	9.3440040276	4.1061078144
4	8.1851105244	3.7431700492	9.5201882548	3.7594888052	9.3763319652	3.8164326820
5	8.4592147545	3.3460084781	9.9038244602	3.4602903181	9.6519009702	3.4992076079
6	9.0673116818	2.9449889133	10.6018768364	3.1325900058	10.2060033528	3.1440596831
7	10.0557266500	2.5197980734	11.7007294372	2.7604692418	11.1093962855	2.7374992954
8	11.5303031948	2.0484433716	13.3574883436	2.3201655255	12.4871637697	2.2605946027
9	13.6800933636	1.5026879752	15.8577984930	1.7728678617	14.5571182701	1.6844954512
10	16.8306930352	0.8415968385	19.7510850824	1.0482398348	17.7109254612	0.9609642283
11	21.5547425985	0.0000000000	26.2132226347	0.0000000000	22.7016008493	0.0000000000

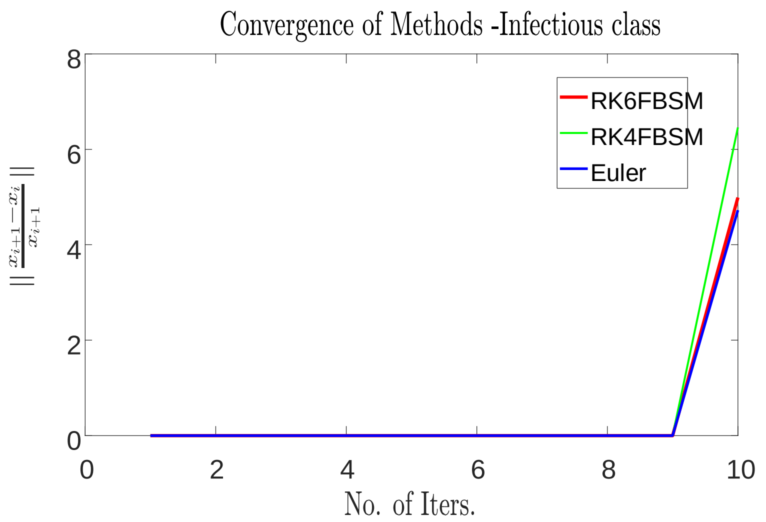


Figure 2: Rate of convergence in 10 iterations.

Example 3: We considered the model on optimal control and comprehensive cost effectiveness analysis for COVID-19 [2]

$$J(u_1,u_2,u_3,u_4) := \min \int_0^T \left[A_1E + A_2I + A_3A + A_4B + \frac{1}{2}\sum_{i=1}^{4} D_i u_i^2(t) \right] dt \tag{40}$$

Subject to the non-autonomous system below

$$\begin{cases} \frac{dS}{dt} &= \Lambda - (1-u_1(t))\left(\frac{\beta_1E+\beta_2I+\beta_3A}{N}\right)S - (1-u_1(t)-u_2(t))\frac{\beta_4B}{N}S - dS, \\ \frac{dE}{dt} &= (1-u_1(t))\left(\frac{\beta_1E+\beta_2I+\beta_3A}{N}\right)S + (1-u_1(t)-u_2(t))\frac{\beta_4B}{N}S - (\delta+d)E, \\ \frac{dI}{dt} &= (1-\tau)\delta E - (d+d_1+\gamma_1)I, \\ \frac{dA}{dt} &= \tau\delta E - (d+\gamma_2)A, \\ \frac{dR}{dt} &= \gamma_1 I + \gamma_2 A - dR, \\ \frac{dB}{dt} &= (1-u_3(t))(\psi_1E+\psi_2I+\psi_3A) - (u_4(t)+\phi)B, \end{cases} \tag{41}$$

where $A_i > 0 (i = 1,2,3,4)$. The derived adjoint equations were

$$\begin{cases} \frac{d\lambda_1}{dt} = (\lambda_1-\lambda_2)((1-u_1)(\frac{\beta_1E^*+\beta_2I^*+\beta_3A^*}{N^2})(E^*+I^*+A^*+R^*))+ \\ (\lambda_1-\lambda_2)(1-u_1-u_2)\beta_4\frac{B^*}{N^2}(E^*+I^*+A^*+R^*) + \lambda_1 d, \\ \frac{d\lambda_2}{dt} = -A_1 + (\lambda_1-\lambda_2)(1-u_1)S^*(\frac{S^*+I^*+A^*+R^*)\beta_1-(\beta_2I^*+\beta_3A^*)}{N^2})+ \\ (\lambda_2-\lambda_1)(1-u_1-u_2)\beta_4\frac{B^*S^*}{N^2} + (\delta+d)\lambda_2 - \lambda_2(1-\tau)\delta\lambda_3 - \tau\delta\lambda_4 - (1-u_3)\psi_1\lambda_6 \\ \frac{d\lambda_3}{dt} = -A_2 + (\lambda_1-\lambda_2)(1-u_1)S^*(\frac{(S^*+E^*+A^*+R^*)\beta_2-(\beta_1E^*+\beta_3A^*)}{N^2})+ \\ (\lambda_2-\lambda_1)(1-u_1-u_2)\beta_4\frac{B^*S^*}{N^2} + (\delta+d_1+\gamma_1)\lambda_3 - \gamma_1\lambda_5 - (1-u_3)\psi_2\lambda_6, \\ \frac{d\lambda_4}{dt} = -A_3 + (\lambda_1-\lambda_2)(1-u_1)S^*(\frac{(S^*+E^*+I^*+R^*)\beta_3-(\beta_1E^*+\beta_2I^*)}{N^2})+ \\ (\lambda_2-\lambda_1)(1-u_1-u_2)\beta_4\frac{B^*S^*}{N^2} + (d_1+\gamma_2)\lambda_4 - \gamma_2\lambda_5 - (1-u_3)\psi_3\lambda_6, \\ \frac{d\lambda_5}{dt} = (\lambda_2-\lambda_1)(1-u_1)(\frac{(\beta_1E^*+\beta_2I^*+\beta_3A^*)S^*}{N^2})+ \\ (\lambda_2-\lambda_1)(1-u_1-u_2)\beta_4\frac{S^*}{N^2} + \lambda_5 d, \\ \frac{d\lambda_6}{dt} = -A_4 + (\lambda_1-\lambda_2)(1-u_1-u_2)\beta_4\frac{S^*}{N} + (u_4+\phi)\lambda_6 d, \end{cases} \tag{42}$$

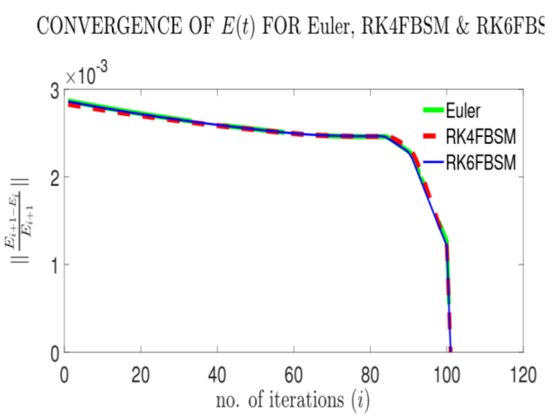
while the optimal control characterizations were;

$$u_1^*(t) = \min\left\{\max\left\{0, \frac{(\lambda_2-\lambda_1)(\beta_1 E^* + \beta_2 I^* + \beta_3 A^* + \beta_4 B^*)S}{D_1 N}\right\}, u_{1\max}\right\}$$
$$u_2^*(t) = \min\left\{\max\left\{0, \frac{(\lambda_2-\lambda_1)\beta_4 B^* S^*}{D_2 N}\right\}, u_{2\max}\right\},$$
$$u_3^*(t) = \min\left\{\max\left\{0, \frac{\lambda_6(\psi_1 E^* + \psi_2 I^* + \psi_3 A^*)}{D_3}\right\}, u_{3\max}\right\},$$
$$u_4^*(t) = \min\left\{\max\left\{0, \frac{\phi B^* \lambda_6}{D_4}\right\}, u_{4\max}\right\}. \tag{43}$$

Simulating with the following parameters $\beta_1 = 0.1233; \beta_2 = 0.0542; \beta_3 = 0.0020; \beta_4 = 0.1101; \delta = 0.1980; \tau = 0.3085; d = 1/(74.87 * 365); d_1 = 0.0104; \gamma_1 = 0.3680; \gamma_2 = 0; \psi_1 = 0.2574; \psi_2 = 0.2798; \psi_3 = 0.1584; \phi = 0.3820$ yields the results below.

Table 3: Convergence analysis of State variable ($E(t)$) for example 3

S/N	Euler		RK4FBSM		Proposed RK6FBSM	
	E_E	$\lVert\frac{E_{i+1}-E_i}{E_{i+1}}\rVert$	$EK4$	$\lVert\frac{E_{i+1}-E_i}{E_{i+1}}\rVert$	$EK6$	$\lVert\frac{E_{i+1}-E_i}{E_{i+1}}\rVert$
1	1.5000000000	-	1.5000000000		1.5000000000	
11	1.4586854415	2.7646377×10^{-3}	1.4582818367	2.7890189×10^{-3}	1.4581829510	2.7953366×10^{-3}
21	1.4194850125	2.6987961×10^{-3}	1.4187988721	2.7167244×10^{-3}	1.4186243717	2.7216273×10^{-3}
31	1.3821955578	2.6394372×10^{-3}	1.3813219909	2.6520393×10^{-3}	1.3810945185	2.6556630×10^{-3}
41	1.3466385402	2.5859736×10^{-3}	1.3456514066	2.5942269×10^{-3}	1.3453892655	2.5968128×10^{-3}
51	1.3126570130	2.5378496×10^{-3}	1.3116129021	2.5425930×10^{-3}	1.3113294894	2.5443931×10^{-3}
61	1.2801128162	2.4946384×10^{-3}	1.2790541427	2.4967666×10^{-3}	1.2787595421	2.4977039×10^{-3}
71	1.2488168407	2.4685349×10^{-3}	1.2477535795	2.4713852×10^{-3}	1.2474913995	2.4675198×10^{-3}
81	1.2184644912	2.4616005×10^{-3}	1.2173844274	2.4656931×10^{-3}	1.2171894367	2.4600497×10^{-3}
91	1.1893046217	2.3341404×10^{-3}	1.1885613052	2.2797627×10^{-3}	1.1883097475	2.2930852×10^{-3}
101	1.1685184378	1.2045593×10^{-3}	1.1684734094	1.2114794×10^{-3}	1.1676305020	1.2714446×10^{-3}



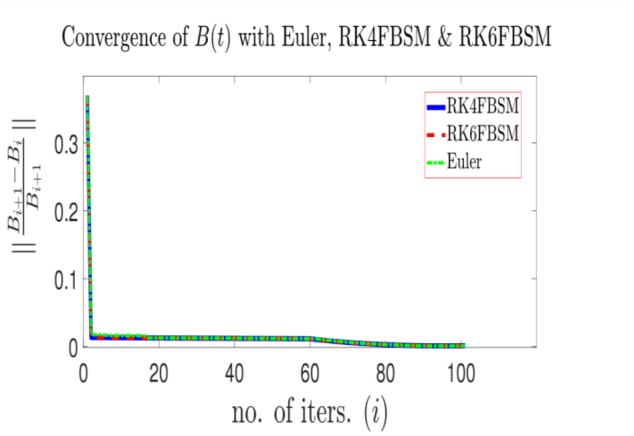


Fig. 3: Convergence of $E(t)$ in 101 iterations. Fig 4: Convergence of $B(t)$ in 101 iterations.

DISCUSSION OF RESULTS

In example 1, the rate of convergence of the 3 methods: Euler, RK4 and Rk6 were compared on the state variable as demonstrated on table 1. It was discovered that the rate of convergence of the RK6 compares favorably with RK4 with higher level of accuracy after 10 iterations. In similar manner, Table 2 and 3 were used to compare the Iterates for the Euler, RK4 and RK6 forward-backward sweep methods on the state variables of examples 2 and 3 respectively. Figures 2, 3 and 4 were used to illustrate the rate of convergences which shows that the RK6FBSM performs excellently well although the computational efforts is more in terms of rigors of coding and process time.

CONCLUSION

The adaptation of the 6th order Runge-Kutta forward-backward sweep algorithm for solving generalized optimal control problems with bounded control arrives at an accurate result at a faster rate of convergence compared to the Runge-Kutta of order four (RK4), due to its stability and higher numerical order of convergence. This adaptation is essential for handling mathematical models with large number of non-linear dynamical equations. Therefore, the sixth order Runge-Kutta forward-backward sweep algorithm seeks to provide a more effective and efficient method due to its speed, accuracy, higher rate of convergence, suitability and versatility for real-time or practical applications such as the Epidemiological and general Biomedical models (see MATLAB code in Appendix).

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