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EXACT SOLUTION FOR TWO-DIMENSIONAL FLOW THROUGH CHANNELS WITH A PLANE PERMEABLE BOUNDARY BETWEEN TWO CHAMBERS WITH UNIFORM SUCTION/INJECTION

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ABSTRACT

A one-dimensional model to determine the laminar ow of a fluid in a porous channel with wall suction or injection is proposed. The approach is based on the integration of the Navier–Stokes equations using the analytical solutions for the two-dimensional local velocity and pressure fields obtained from the asymptotic developments at low filtration Reynolds number proposed by Berman [1] and Yuan and Finkelstein [2]. It is noticeable that the resulting one-dimensional model preserves the whole ow properties, in particular the inertial terms which can affect the wall suction conditions. The model is validated in the case of a single porous channel of rectangular or circular cross-section with uniform or variable wall suction. Then the model is applied to a two-dimensional multi-channel system which consists of a great number of adjacent entrance and exit channels connected by a filter porous medium. All existing models aren't analytical, and need to use complex numerous calculations. The present model is a first an attempt to reduce the problem to a simple analytical scheme based on Berman Similarity and perturbation series solution method that allows it to be used by general engineers not using complex mathematical methods.

Keywords: laminar flow, filtration, porous media, porous channel, suction, ejection

INTRODUCTION

Numerous filtration systems consist of parallel porous channel bundles, i.e., multi-channel systems. Membrane modules used in microfiltration and ultrafiltration are typically multi-channel systems[3,4] Flat-plate membrane modules, which are the earliest configurations developed for commercial applications, use multiple at sheet membranes in a sandwich arrangement consisting of the support plate, the membrane and the channel spacer. The hollow fiber modules consist of an array of narrow-bore fibers with a dense skin layer at the lumen side of the fiber and a macro-porous matrix for rigidity. The multi-channel tubular devices are made of individual porous tubes, which support the membranes, placed inside a sleeve to form a single tube cartridge. The pleated filters are also multi-channel filtration systems developed to arrange large plane filtration area on relatively small base areas [3, 4]

All the models discussed above for laminar ow in a porous channel with suction apply to multi-channel systems if there is no coupling between the individual channels.
This statement is incorrect if there is coupling. At this stage, some refinements are required to include the coupling in the models. The problem can be considered as coupling of two separate problems; (i) laminar ow in a porous channel with wall suction, (ii) laminar ow in a porous channel with wall injection.
In addition, the common porous wall of the two channels is shared with specific porous media characteristics. Thus, a number of models were developed to investigate fluid ow in a unit element of a multi-channel filtration system which consists of two coupled channels [5 – 7]. However, these models still fail in the case of multi-channel systems with spatial heterogeneities such as spatial distribution of entrance ow rates, spatial distribution of channel width, or unexpected plugging of some entrance channels for instance. In this latter case, the flow should be modeled for the entire system to account for the complex geometry and boundary conditions.
Moreover, all existing models aren't analytical, and need to use complex numerous calculations. So, it is a first an attempt to reduce the problem to a simple analytical scheme that allows it to be used by general engineers not using complex mathematical methods.

Description of the Model

This study will focus on 2-D fluid flow through channels, where the plane boundary between two chambers is permeable with uniform suction/injection.
The main assumption is that the both chambers are completely identical with the only difference that the porous wall is the injection boundary for the first chamber, and for the second one it is the boundary of suction. So we may consider the flow in the right chamber at the scheme shown at the Fig. 1; it is shown at the Fig. 2. The left chamber has a similar view with the same coordinate system where in the both cases, for the both chambers, the y-axis is directed towards the porous wall. Thus the both flows are considered in independent coordinate systems, and they are connected only by a common porous boundary.

Note, that the channel of the left camber may have another width $h = h_2$. The channel widths $h = h_1,\ h_2$ are assumed to be constants. As shown in the problem geometry below, u and v are the velocity components which are the functions of x and y, μ is the dynamic viscosity. For the model problem under investigation, we make the following additional assumptions:

(i) Formulate a mathematical model which determines the nature or behavior of the steady laminar flow in the channel with one porous wall.
(ii) The fluid is viscous and incompressible.
(iii) A two dimensional flow scenario is considered.
(iv) The flow is driven by combined action of wall suction/injection and pressure gradient.
(v) A steady state flow situation is considered.

Under the above assumptions, the model equations of motion above reduce to:

$$\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\\
&u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),\\
&u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).
\end{aligned} \tag{1.1}$$

The boundary conditions may be written in the form

$$u\big|_{y=h} = 0,\ v\big|_{y=h} = \pm V, \tag{1.2a}$$

$$u\big|_{y=0} = 0,\ v\big|_{y=0} = 0, \tag{1.2b}$$

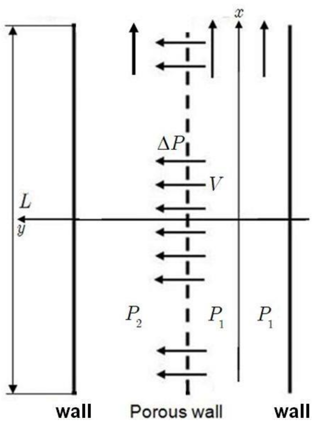

Figure 1: The total Scheme

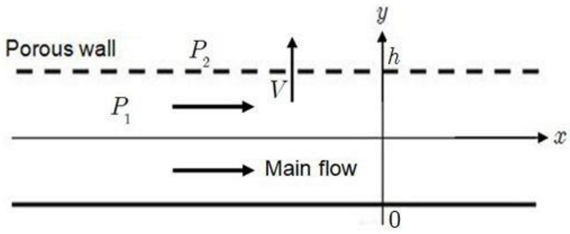

Figure 2: Identical scheme for the both, left and right chambers.

The initial velocity $\pm V$ at the porous wall ($+1$ is for injection, and -1 is for suction), at $y = h$, may be found from,

$$V = \frac{k}{\mu}\frac{\partial P}{\partial y}\bigg|_{y=h} \approx \frac{k}{\mu}\frac{\Delta P}{\delta} = \frac{k}{\mu}\frac{P_1 - P_2}{\delta} \tag{1.3}$$

Where, $k\ [m^2]$ is the coefficient of permeability, and δ is the width of porous wall. We then write the equations of motion in a dimensionless form, starting by scaling the variables as follows:

$$(x, y)/h = (x^*, y^*),\ (u, v)/V = (u^*, v^*),\ P/(\mu V/h) = P^*, \tag{1.4a}$$
$$\partial/\partial x \Rightarrow h^{-1}\partial/\partial x^*,\ \partial/\partial y \Rightarrow h^{-1}\partial/\partial y^* \tag{1.4b}$$

The quantities denoted by an asterisk are in non-dimensional form. Substituting the non-dimensional quantities (as shown in (1.4a) – (1.4b) above) in the equations of motion (1.1) it will change them to:

$$\begin{aligned}
&\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0,\\
&\frac{V^2}{h}\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{1}{\rho}\frac{\partial}{h\partial x^*}\left(\frac{P^*\mu V}{h}\right) + \frac{\mu V}{\rho h^2}\nabla^2 u^*,\\
&\frac{V^2}{h}\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial}{h\partial x^*}\left(\frac{P^*\mu V}{h}\right) + \frac{\mu V}{\rho h^2}\nabla^2 v^*.
\end{aligned} \tag{1.5}$$

Dividing through by V^2/h and follow to $\mathrm{Re} \square \frac{\rho h V}{\mu}$ is the Reynolds number Re we obtain

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0,$$
$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial}{\partial x^*}\left(p^*\right)+\frac{1}{\mathrm{Re}}\nabla^2 u^*,$$
$$u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y}=-\frac{\partial}{\partial x^*}\left(p^*\right)+\frac{1}{\mathrm{Re}}\nabla^2 v^*, \tag{1.6}$$

Where, $p^*=P^*/\ \mathrm{Re}$. Dimensionless form for boundary conditions is form

$$u^*\Big|_{y^*=1}=0,\ v^*\Big|_{y^*=1}=\pm 1 \tag{1.7a}$$

$$u^*\Big|_{y^*=0}=0,\ v^*\Big|_{y^*=0}=0 \tag{1.7b}$$

Next, we will omit the asterisks (*) at the superscripts; here $+1$ is for injection, and -1 – for suction.

Stream function

We can define the stream function ψ for two dimensional flow by expressing the flow velocity as,

$$\mathbf{u}=(u,v,0)=\nabla\times\boldsymbol{\psi} \tag{2.1}$$

Where, $\boldsymbol{\psi}=(0,0,\psi)$. In Cartesian coordinate system this is equivalent to

$$u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x} \tag{2.2}$$

Then, the continuity equation is satisfied identically.

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right)=0\ \Rightarrow\ \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0 \tag{2.3}$$

The total derivative is

$$d\psi=\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy=-vdx+udy \tag{2.4}$$

Whereby $\psi(x,y)$ is a constant along the streamline, then $d\psi=0$. Hence, the equation (2.4) may be rewritten as

$$\frac{dx}{u}=\frac{dy}{v} \tag{2.5}$$

This is the equation which we use to determine a streamline. Streamlines are therefore lines of constant and they cannot cross each other except at stagnant points.

Moreover, we can define a stream function by modifying the Navier-Stokes equations. This can be easily done by differentiating the x-momentum equation of the system (1.1) with respect to y and the y-momentum equation (1.1) with respect to x and then subtract, so as to eliminate the pressure term. The resulting equation becomes:

$$\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi)-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi)-\frac{1}{\mathrm{Re}}(\nabla^4\psi)=0 \tag{2.6}$$

where ∇^4 is the 2-D bi-harmonic operator. The associated boundary conditions are

(i) On $y=0$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right)=\frac{\partial^2\psi}{\partial y^2},\ \frac{\partial v}{\partial x}=\frac{\partial}{\partial x}\left(-\frac{\partial\psi}{\partial x}\right)=-\frac{\partial^2\psi}{\partial x^2},$$
$$\left.\frac{\partial^2\psi}{\partial y^2}\right|_{y=0}=\left.\psi'_y\right|_{y=0}=0,\ \left.\frac{\partial^2\psi}{\partial x^2}\right|_{y=0}=\left.\psi'_x\right|_{y=0}=0 \tag{2.7a}$$

(ii) On $y=1$:

$$\left.u\right|_{y=1}=\left.\psi'_y\right|_{y=1}=0,\ \left.v\right|_{y=1}=\left.\psi'_x\right|_{y=1}=\pm1 , \tag{2.7b}$$

$(+1)$ is for injection, and (-1) – for suction.

Berman Similarity

Consider the Berman problem [8 – 9] where a two-dimensional flow in a channel is considered and $x\in(-\infty,\infty),\ y\in(0,h)$. For an incompressible steady-state flow, we consider a stream function in terms (2.2). Using Jacobian determinant, our motion expression (2.6) written in terms of stream function may be rewritten as,

$$\frac{\partial(\nabla^2\psi,\psi)}{\partial(x,y)}\equiv\begin{vmatrix}\frac{\partial}{\partial x}(\nabla^2\psi) & \frac{\partial}{\partial y}(\nabla^2\psi)\\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y}\end{vmatrix}=\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi)-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi)=\frac{1}{\mathrm{Re}}(\nabla^4\psi) \tag{2.8}$$

Suppose $\Omega=\nabla^2\psi$, equation (2.8 becomes)

$$\frac{\partial(\Omega,\psi)}{\partial(x,y)}=\frac{1}{\mathrm{Re}}(\nabla^2\Omega)\,. \tag{2.9}$$

Where, $\Omega=\nabla^2\psi$ is the velocity. We seek for similarity solution of the form $\psi=xf(\eta)$, where x,y are all dimensionless, and $y=\eta$. Then,

$$\psi'_x=f(\eta),\ \ \psi'_\eta=xf'_\eta,\ \ \psi''_{\eta\eta}=xf''_{\eta\eta}\,.$$

Applying this to equation (2.8) it yields

$$f'f''_{\eta\eta}-ff'''_{\eta\eta\eta}=\mathrm{Re}^{-1}f^{IV}{}_{\eta\eta\eta\eta} \tag{2.10}$$

with boundary conditions

(i) At $\eta=0$

$$\left.\psi'_x\right|_{y=0}=0\ \Rightarrow\ f(0)=0;\ \ \left.\psi''_{yy}\right|_{y=0}=\left.\psi'_y\right|_{y=0}=0\ \Rightarrow\ f'_\eta(0)=f''_{\eta\eta}(0)=0\,. \tag{2.11a}$$

(ii) At $\eta=1$

$$\left.\psi'_y\right|_{y=1}=0\ \Rightarrow\ f'(1)=0,\ \ \left.\psi'_x\right|_{y=1}=\pm1\ \Rightarrow\ f(1)=\pm1 \tag{2.11b}$$

due to $\psi = xf(\eta),\ \psi'_x = f(\eta),\ \psi'_\eta = xf'_\eta,\ \psi''_{\eta\eta} = xf''_{\eta\eta}$.
As a result we obtain the boundary value problem

$$\mathrm{Re}(f'f'' - ff''') = f^{IV} \tag{2.12a}$$

$$f(0) = 0,\ f'(0) = 0,\ f(1) = \pm 1,\ f'(1) = 0 \tag{2.12b}$$

Pressure gradient

In many fluid flow problems we consider the speed of the flow at any point to be proportional to the change of pressure per unit length and this is what we call the pressure gradient. This implies that the flow is always in the direction of decreasing pressure. An adverse pressure gradient occurs when the static pressure increases in the direction of the flow. Mathematically this is expressed as $dP/dx > 0$. The pressure gradient is said to be favorable to the flow when $\frac{dP}{dx} < 0$. Since the fluid in the inner part of the boundary layer is relatively slower, it is more greatly affected by the increasing pressure gradient. For a large enough pressure increase, this fluid may slow to zero velocity or even become reversed. When flow reversal occurs, the flow is said to be separated from the surface.

Pressure gradient is one of the factors that influence a flow immensely and the shear stress caused by viscosity has a retarding effect upon the flow. This effect can however be overcome if there is a negative pressure gradient offered to the flow. A negative pressure gradient is termed a favorable pressure gradient since it enables the flow. A positive pressure gradient has the opposite effect and is termed the Adverse Pressure Gradient.
From the second equation of the system equation (1.2), we have

$$-\frac{\partial p}{\partial x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{1}{\mathrm{Re}}\nabla^2 u\ . \tag{3.1}$$

Substituting

$$uu'_x = \psi'_\eta(\psi'_\eta)'_x = (xf'_\eta)(f'_\eta) = xf'^2_\eta, \quad vu'_y = -\psi'_x(\psi''_{\eta\eta}) = -(f)(xf''_{\eta\eta}) = -xff''_{\eta\eta},$$
$$\nabla^2 u = u''_{xx} + u''_{yy} = ((\psi'_\eta)'_x)'_x + ((\psi''_{\eta\eta}))'_\eta = ((xf''_{\eta\eta}))'_\eta = x\psi'''_{\eta\eta\eta} \tag{3.2}$$

Gives

$$-(\partial p/\partial x)\,\mathrm{Re} = x\left[\mathrm{Re}(f'^2_\eta - ff''_{\eta\eta}) - f'''_{\eta\eta\eta}\right]. \tag{3.3}$$

Let us assume $-\mathrm{Re}\frac{\partial p}{dx} = xA$ with A to be the pressure gradient constant. So equation (3.3) has a view,

$$A = \mathrm{Re}(f'^2_\eta - ff''_{\eta\eta}) - f'''_{\eta\eta\eta} \tag{3.4}$$

Remember, that $p = P/\mathrm{Re}$, one obtains $-\frac{\partial P}{dx} = xA/\mathrm{Re}$, and finding A we will find $\frac{\partial P}{dx}$ and so

$$P(x) = P_0 - \tfrac{1}{2}A(x)^2 = P_0 - \tfrac{1}{2}A(x_{\text{dimensional}}/h)^2 \equiv P_0 - \Delta P\ . \tag{3.5}$$

The most researches try to resolve the equation (3.4) with the boundary conditions (2.12b). Nevertheless, attempts to solve this problem, in contrast with solution of the classical Berman problem with initial boundary conditions [8, 9], directly applying the perturbation method, do not lead to success since the boundary value problem turns out to be incorrect. The reason is that symmetrical problem is connected with the flow between the both porous walls when we have a deal with initial boundary conditions. Note,

that numerous attempts to solve this boundary value problem by reducing it to the classical Berman problem using the method presented by Terrill [9 – 11] and improved by Cox [12, 13] turns out to be a rather complicated matter.
The solution of the boundary value problem (2.12a) – (2.12b), as it will be shown below, using the well-known perturbation method, does not require significant difficulties.

Perturbation series solution method

The flow of an incompressible Newtonian fluid through a rectangular micro tube is considered, with x-axis being in the axial direction as shown in the problem geometry.
By seeking similarity solution of the form $\psi = xVf(\eta)$ and $\eta = y/h$ we have shown that the Navier-Stokes equations can be reduced to (compare with (2.12)):

$$f^{IV} + \mathrm{Re}(ff''' - f'f'') = 0, \tag{4.1a}$$
$$f(0) = 0,\ f'(0) = 0,\ f(1) = \pm 1,\ f'(1) = 0. \tag{4.1b}$$

Equation (4.1a) can be solved by perturbation series method, by seeking the solution in the form of power series in Re. That is,

$$f(\eta) = \sum_{k=0}^{\infty} f_k(\eta)\,\mathrm{Re}^k \tag{4.2}$$

For small Re (let say for the first two terms), the higher powers of Re such as $\mathrm{Re}^2, \mathrm{Re}^3, \ldots, \mathrm{Re}^n, \ldots$ will also give us very small values of Re, therefore, they can be neglected or approximated to zero. Hence, equation (4.2) above reduces to:

$$f(\eta) = f_0 + f_1 \cdot \mathrm{Re} + f_2 \cdot \mathrm{Re}^2 + O(\mathrm{Re}^3), \tag{4.3}$$

which implies that those which are in higher powers or Re are negligible. Introducing (4.2) into (4.1a), equating the coefficients at the same Reynolds numbers degrees, and after neglecting the terms with powers $n \geq 3$,

$$\begin{aligned} &f_0^{IV} + f_1^{IV}\,\mathrm{Re} + f_1^{IV}\,\mathrm{Re}^2 + \mathrm{Re}(f_0 + f_1\,\mathrm{Re})(f_0''' + f_1'''\mathrm{Re}) - \\ &\quad - \mathrm{Re}(f_0' + f_1'\mathrm{Re})(f_0'' + f_1''\mathrm{Re}) = f_0^{IV} + f_1^{IV}\,\mathrm{Re} + \mathrm{Re}(f_0 f_0''' - f_0' f_0'') \\ &\quad + \mathrm{Re}^2\left[(f_1 f_0''' + f_0 f_1''') - (f_1' f_0'' + f_0' f_1'')\right] = 0, \end{aligned}$$

We obtain

$$\begin{aligned} &f_0^{IV} = 0, \\ &f_1^{IV} + f_0 f_0''' - f_0' f_0'' = 0, \\ &f_2^{IV} + (f_1 f_1''' - f_1' f_1'') + (f_1 f_0''' - f_0' f_1'') = 0, \ldots \end{aligned} \tag{4.4}$$

Zeroth order equation: When we substitute solution (4.3) in equation (4.1a) and collecting the coefficients of like powers of Re, it will reduce to the zeroth and first order equations of the form:

$$f_0^{IV} = 0, \tag{4.5a}$$

with the boundary conditions

$$f_0(0),\ f_0'(0)=0,\ f_0(1)=\pm 1,\ f_0'(1)=0\,, \tag{4.5b}$$

Where, the upper and bottom signs "$\pm$" or "$\mp$", there and bellow, correspond to the both cases, “injection” (upper signs) and “suction” (bottom signs), accordingly. Integrating the equation (4.5a) gives

$$\begin{aligned}
&f_0'''=C_0,\ f_0''=C_0\eta+C_1,\\
&f_0'=\tfrac{1}{2}C_0\eta^2+C_1\eta+C_2,\\
&f_0=\tfrac{1}{6}C_0\eta^3+\tfrac{1}{2}C_1\eta^2+C_2\eta+C_3
\end{aligned} \tag{4.6}$$

Using the boundary conditions (4.5a) give algebraic equations for C_0, C_1, C_2, C_3, which define coefficients of integration

$$\begin{aligned}
&f_0(0)=0\ \Rightarrow\ C_3=0\\
&f_0'(0)=0\ \Rightarrow\ C_2=0\\
&f_0(1)=\pm 1\ \Rightarrow\ \tfrac{1}{6}C_0+\tfrac{1}{2}C_1+C_2+C_3=\pm 1\\
&f_0'(1)=0\ \Rightarrow\ \tfrac{1}{2}C_0+C_1+C_2=0.
\end{aligned} \tag{4.7}$$

As a result $C_0=\mp 12,\ C_1=\pm 6,\ C_2=C_3=0$, and hence,

$$f_0=\mp(2\eta^3-3\eta^2),\quad f_0'=\mp 6(\eta^2-\eta),\quad f_0''=\mp 6(2\eta-1),\quad f_0'''=\mp 12 \tag{4.8}$$

Note, that $f_0=-2\eta^3+3\eta^2$ is for injection, and $f_0=2\eta^3-3\eta^2$ is for suction.

First order equation: The solution f_1 may be found from the problem (4.4b) – (4.5b),

$$\begin{aligned}
&f_1^{IV}+f_0f_0'''-f_0'f_0''=0\\
&f_1(0)=f_1'(0)=f_1(1)=f_1'(1)=0.
\end{aligned}$$

Substitution f_0 from (4.8) gives,

$$\begin{aligned}
f_0f_0'''-f_0'f_0''&=\mp(2\eta^3-3\eta^2)(\mp 12)-36(\eta^2-\eta)(2\eta-1)=\\
&=12(2\eta^3-3\eta^2)-36(2\eta^3-3\eta^2+\eta)=\\
&=24\eta^3-36\eta^2-72\eta^3+108\eta^2-36\eta=-12(4\eta^3-6\eta^2+3\eta).
\end{aligned}$$

Hence,

$$f_1^{IV}=-(f_0f_0'''-f_0'f'')=12(4\eta^3-6\eta^2+3\eta)\,, \tag{4.9a}$$

$$f_1(0)=f_1'(0)=f_1(1)=f_1'(1)=0. \tag{4.9b}$$

Integration the equation (4.9) gives,

$$\begin{aligned}
&f_1'''=12(\eta^4-2\eta^3+\tfrac{3}{2}\eta^2)+C_0,\\
&f_1''=12(\tfrac{1}{5}\eta^5-\tfrac{1}{2}\eta^4+\tfrac{1}{2}\eta^3)+C_0\eta+C_1,\\
&f_1'=12(\tfrac{1}{30}\eta^6-\tfrac{1}{10}\eta^5+\tfrac{1}{8}\eta^4)+\tfrac{1}{2}C_0\eta^2+C_1\eta+C_2,\\
&f_1=12(\tfrac{1}{210}\eta^7-\tfrac{1}{60}\eta^6+\tfrac{1}{40}\eta^5)+\tfrac{1}{6}C_0\eta^3+\tfrac{1}{2}C_1\eta^2+C_2\eta+C_3,
\end{aligned} \tag{4.10}$$

Using the boundary conditions from (4.9a) gives algebraic equations for C_0, C_1, C_2, C_3, which define coefficients of integration

$$f_1(0)=0 \;\Rightarrow\; C_3=0, \quad f_1'(0)=0 \;\Rightarrow\; C_2=0,$$
$$f_1(1)=0 \;\Rightarrow\; \tfrac{11}{70}+\tfrac{7}{360}C_0+\tfrac{1}{2}C_1=0, \quad f_1'(1)=0 \;\Rightarrow\; \tfrac{7}{10}+\tfrac{1}{2}C_0+C_1=0. \tag{4.11}$$

Hence,

$$\tfrac{7}{180}C_0+C_1=-\tfrac{11}{70},\; C_0+2C_1=-\tfrac{7}{5}.$$

As a result we obtain $C_0=-\frac{19}{7}\frac{83}{9}=-\frac{1577}{63},\; C_1=-\frac{7}{10}+\frac{1577}{126}=\frac{-441+7885}{630}=\frac{3722}{315}$, and so

$$f_1=\tfrac{2}{35}\eta^7-\tfrac{1}{5}\eta^6+\tfrac{3}{10}\eta^5-\tfrac{1577}{378}\eta^3+\tfrac{1861}{315}\eta^2. \tag{4.12}$$

Second order equation: The solution f_1 may be found from the problem (4.4b) – (4.5b),

$$f_2^{IV}+f_1f_1'''-f_1'f_1''+f_1f_0'''-f_0'f_1''=0$$
$$f_2(0)=f_2'(0)=f_2(1)=f_2'(1)=0.$$

Substitution f_0 from (4.8) and f_1 from (4.12) gives,

$$\begin{aligned}
f_1f_1'''-f_1'f_1''=&\left[\tfrac{1}{5}\left(\tfrac{2}{7}\eta^7-\eta^6+\tfrac{3}{2}\eta^5\right)-\tfrac{1577}{378}\eta^3+\tfrac{1861}{315}\eta^2\right]\cdot\left[\tfrac{6}{5}(10\eta^4-20\eta^3+15\eta^2)-\tfrac{1577}{63}\right]-\\
&-\left[\tfrac{6}{5}\left(\tfrac{1}{3}\eta^6-\eta^5+\tfrac{5}{4}\eta^4\right)-\tfrac{1577}{126}\eta^2+\tfrac{3722}{315}\eta\right]\cdot\left[\tfrac{6}{5}(2\eta^5-5\eta^4+5\eta^3)-\tfrac{1577}{63}\eta+\tfrac{3722}{315}\right]=\\
&=-\left(\tfrac{48}{175}\eta^{11}-\tfrac{264}{175}\eta^{10}+\tfrac{132}{35}\eta^9-\tfrac{306}{10}\eta^8+\tfrac{1212}{189}\eta^7+\tfrac{10103}{3155}\eta^6-\tfrac{1577}{21}\eta^5+\tfrac{14888}{105}\eta^4+\right.\\
&\left.+\tfrac{2486929}{7938}\eta^3-\tfrac{2934797}{6615}\eta^2+\tfrac{13853284}{99225}\eta\right),
\end{aligned} \tag{4.13a}$$

$$\begin{aligned}
&f_1f_0'''-f_0'f_1''=\mp 12\left[\tfrac{1}{5}\left(\tfrac{2}{7}\eta^7-\eta^6+\tfrac{3}{2}\eta^5\right)-\tfrac{1577}{378}\eta^3+\tfrac{1861}{315}\eta^2\right]\pm\\
&\pm 6(\eta^2-\eta)\left[\tfrac{6}{5}(2\eta^5-5\eta^4+5\eta^3)-\tfrac{1577}{63}\eta+\tfrac{3722}{315}\right]=\pm 6\left\{\left[\tfrac{2}{5}\left(-\tfrac{2}{7}\eta^7+\eta^6-\tfrac{3}{2}\eta^5\right)+\tfrac{1577}{189}\eta^3-\tfrac{3722}{315}\eta^2\right]+\right.\\
&\left.(\eta^2-\eta)\left[\tfrac{6}{5}(2\eta^5-5\eta^4+5\eta^3)-\tfrac{1577}{63}\eta+\tfrac{3722}{315}\right]\right\}=\pm 6\left(\tfrac{16}{7}\eta^7-8\eta^6+\tfrac{57}{5}\eta^5-6\eta^4+\tfrac{1577}{63}\eta^2-\tfrac{3722}{315}\eta\right).
\end{aligned} \tag{4.13b}$$

Follow to

$$\begin{aligned}
&f_1=\tfrac{2}{35}\eta^7-\tfrac{1}{5}\eta^6+\tfrac{3}{10}\eta^5-\tfrac{1577}{378}\eta^3+\tfrac{1861}{315}\eta^2, \quad f_1'=\tfrac{2}{5}\eta^6-\tfrac{6}{5}\eta^5+\tfrac{3}{2}\eta^4-\tfrac{1577}{126}\eta^2+\tfrac{3722}{315}\eta,\\
&f_1''=\tfrac{12}{5}\eta^5-6\eta^4+6\eta^3-\tfrac{1577}{63}\eta+\tfrac{3722}{315}, \qquad f_1'''=12\eta^4-24\eta^3+18\eta^2-\tfrac{1577}{63},
\end{aligned} \tag{4.14}$$

and (4.8) we obtain for injection:

$$\begin{aligned}
f_1f_1'''-f_1'f_1''+f_1f_0'''-f_0'f_1''=&-\left(\tfrac{48}{175}\eta^{11}-\tfrac{264}{175}\eta^{10}+\tfrac{132}{35}\eta^9-\tfrac{306}{10}\eta^8-\tfrac{460}{63}\eta^7+\tfrac{161543}{3155}\eta^6-\right.\\
&\left.-\tfrac{15067}{105}\eta^5+\tfrac{18868}{105}\eta^4+\tfrac{2486929}{7938}\eta^3-\tfrac{3928307}{6615}\eta^2+\tfrac{20887864}{99225}\eta\right),
\end{aligned} \tag{4.15a}$$

and for suction:

$$\begin{aligned}
f_1f_1'''-f_1'f_1''+f_1f_0'''-f_0'f_1''=&-\left(\tfrac{48}{175}\eta^{11}-\tfrac{264}{175}\eta^{10}+\tfrac{132}{35}\eta^9-\tfrac{306}{10}\eta^8+\tfrac{1268}{63}\eta^7-\tfrac{141337}{3155}\eta^6-\right.\\
&\left.-\tfrac{703}{105}\eta^5+\tfrac{11108}{105}\eta^4+\tfrac{2486929}{7938}\eta^3-\tfrac{1941287}{6615}\eta^2+\tfrac{6818704}{99225}\eta\right).
\end{aligned} \tag{4.15b}$$

Hence, we have the next two boundary value problem for the both cases, injection and suction, accordingly

$$f_2^{IV} = \frac{48}{175}\eta^{11} - \frac{264}{175}\eta^{10} + \frac{132}{35}\eta^9 - \frac{306}{10}\eta^8 - \frac{460}{63}\eta^7 + \frac{161543}{3155}\eta^6 - \frac{15067}{105}\eta^5 + \frac{18868}{105}\eta^4 + \\ + \frac{2486929}{7938}\eta^3 - \frac{3928307}{6615}\eta^2 + \frac{20887864}{99225}\eta, \tag{4.16a}$$

$$f_2^{IV} = \frac{48}{175}\eta^{11} - \frac{264}{175}\eta^{10} + \frac{132}{35}\eta^9 - \frac{306}{10}\eta^8 + \frac{1268}{63}\eta^7 - \frac{141337}{3155}\eta^6 - \frac{703}{105}\eta^5 + \frac{11108}{105}\eta^4 + \\ + \frac{2486929}{7938}\eta^3 - \frac{1941287}{6615}\eta^2 + \frac{6818704}{99225}\eta. \tag{4.16b}$$

with the boundary conditions (4.9b).
Integrations the equations (4.16a) and (4.16b) for injection and suction, correspondingly, give,

$$\begin{aligned}
f_2^{IV} &= \tfrac{48}{175}\eta^{11} - \tfrac{264}{175}\eta^{10} + \tfrac{132}{35}\eta^9 - \tfrac{153}{5}\eta^8 - \tfrac{460}{63}\eta^7 + \tfrac{161543}{3155}\eta^6 - \tfrac{15067}{105}\eta^5 + \tfrac{18868}{105}\eta^4 + \\
&\quad + \tfrac{2486929}{7938}\eta^3 - \tfrac{3928307}{6615}\eta^2 + \tfrac{20887864}{99225}\eta, \\
f_2''' &= \tfrac{4}{175}\eta^{12} - \tfrac{24}{175}\eta^{11} + \tfrac{66}{175}\eta^{10} - \tfrac{17}{5}\eta^9 + \tfrac{101}{126}\eta^8 + \tfrac{161543}{22085}\eta^7 - \tfrac{15067}{630}\eta^6 + \tfrac{18868}{525}\eta^5 + \\
&\quad + \tfrac{2486929}{31752}\eta^4 - \tfrac{3928307}{19845}\eta^3 + \tfrac{10443932}{99225}\eta^2 + C_0, \\
f_2'' &= \tfrac{4}{2275}\eta^{13} - \tfrac{12}{175}\eta^{12} + \tfrac{6}{175}\eta^{11} - \tfrac{17}{50}\eta^{10} + \tfrac{101}{1134}\eta^9 + \tfrac{161543}{176680}\eta^8 - \tfrac{15067}{4410}\eta^7 + \tfrac{9434}{1575}\eta^6 + \\
&\quad + \tfrac{2486929}{158760}\eta^5 - \tfrac{3928307}{79380}\eta^4 + \tfrac{10443932}{297675}\eta^3 + C_0\eta + C_1, \\
f_2' &= \tfrac{2}{15925}\eta^{14} - \tfrac{2}{2275}\eta^{13} + \tfrac{1}{350}\eta^{12} - \tfrac{17}{550}\eta^{11} + \tfrac{101}{11340}\eta^{10} + \tfrac{161543}{1590120}\eta^9 - \tfrac{15067}{35280}\eta^8 + \tfrac{9434}{11025}\eta^7 + \\
&\quad + \tfrac{2486929}{952560}\eta^6 - \tfrac{3928307}{396900}\eta^5 + \tfrac{2610983}{297675}\eta^4 + \tfrac{1}{2}C_0\eta^2 + C_1\eta + C_2, \\
f_2 &= \tfrac{2}{79625}\eta^{15} - \tfrac{1}{15925}\eta^{14} + \tfrac{1}{4550}\eta^{13} - \tfrac{17}{6600}\eta^{12} + \tfrac{101}{124740}\eta^{11} + \tfrac{10103}{15901200}\eta^{10} - \tfrac{15067}{317520}\eta^9 + \tfrac{4717}{44100}\eta^8 + \\
&\quad + \tfrac{2486929}{6667920}\eta^7 - \tfrac{2934797}{2381400}\eta^6 + \tfrac{3463321}{1488375}\eta^5 + \tfrac{1}{6}C_0\eta^3 + \tfrac{1}{2}C_1\eta^2 + C_2\eta + C_3;
\end{aligned} \tag{4.18a}$$

$$\begin{aligned}
f_2^{IV} &= \tfrac{48}{175}\eta^{11} - \tfrac{264}{175}\eta^{10} + \tfrac{132}{35}\eta^9 - \tfrac{153}{5}\eta^8 + \tfrac{1268}{63}\eta^7 - \tfrac{141337}{3155}\eta^6 - \tfrac{703}{105}\eta^5 + \tfrac{11108}{105}\eta^4 + \\
&\quad + \tfrac{2486929}{7938}\eta^3 - \tfrac{1941287}{6615}\eta^2 + \tfrac{6818704}{99225}\eta. \\
f_2''' &= \tfrac{4}{175}\eta^{12} - \tfrac{24}{175}\eta^{11} + \tfrac{66}{175}\eta^{10} - \tfrac{17}{5}\eta^9 + \tfrac{317}{126}\eta^8 - \tfrac{20191}{3155}\eta^7 - \tfrac{703}{630}\eta^6 + \tfrac{11108}{525}\eta^5 + \\
&\quad + \tfrac{2486929}{31752}\eta^4 - \tfrac{1941287}{19845}\eta^3 + \tfrac{3409352}{99225}\eta^2 + C_0, \\
f_2'' &= \tfrac{4}{2275}\eta^{13} - \tfrac{2}{175}\eta^{12} + \tfrac{6}{175}\eta^{11} - \tfrac{17}{50}\eta^{10} + \tfrac{317}{1134}\eta^9 - \tfrac{20191}{25240}\eta^8 - \tfrac{703}{4410}\eta^7 + \tfrac{5554}{1575}\eta^6 + \\
&\quad + \tfrac{2486929}{158760}\eta^5 - \tfrac{1941287}{79380}\eta^4 + \tfrac{3409352}{297675}\eta^3 + C_0\eta + C_1, \\
f_2' &= \tfrac{2}{15925}\eta^{14} - \tfrac{2}{2275}\eta^{13} + \tfrac{1}{350}\eta^{12} - \tfrac{17}{550}\eta^{11} + \tfrac{317}{11340}\eta^{10} - \tfrac{20191}{227160}\eta^9 - \tfrac{703}{35280}\eta^8 + \tfrac{5554}{11025}\eta^7 + \\
&\quad + \tfrac{2486929}{952560}\eta^6 - \tfrac{1941287}{396900}\eta^5 + \tfrac{852338}{297675}\eta^4 + \tfrac{1}{2}C_0\eta^2 + C_1\eta + C_2, \\
f_2 &= \tfrac{2}{238875}\eta^{15} - \tfrac{1}{15925}\eta^{14} + \tfrac{1}{4550}\eta^{13} - \tfrac{17}{6600}\eta^{12} + \tfrac{317}{124740}\eta^{11} - \tfrac{20191}{2271600}\eta^{10} - \tfrac{703}{317520}\eta^9 + \tfrac{2777}{44100}\eta^8 + \\
&\quad + \tfrac{2486929}{6667920}\eta^7 - \tfrac{1941287}{2381400}\eta^6 + \tfrac{852338}{1488375}\eta^5 + \tfrac{1}{6}C_0\eta^3 + \tfrac{1}{2}C_1\eta^2 + C_2\eta + C_3.
\end{aligned} \tag{4.18b}$$

Using the boundary conditions from (4.17) we obtain algebraic equations for C_0, C_1, C_2, C_3,

$$f_2(0) = 0 \;\Rightarrow\; C_3 = 0; \qquad f_2'(0) = 0 \;\Rightarrow\; C_2 = 0,$$

for injection :

$$
\begin{aligned}
f_2(1)=0 \;\Rightarrow\; & \tfrac{2}{79625}-\tfrac{1}{15925}+\tfrac{1}{4550}-\tfrac{17}{6600}+\tfrac{101}{124740}+\tfrac{10103}{15901200}-\tfrac{15067}{317520}+\tfrac{4717}{44100}+\\
&+\tfrac{2486929}{6667920}-\tfrac{2934797}{2381400}+\tfrac{3463321}{1488375}+\tfrac{1}{6}C_0+\tfrac{1}{2}C_1+C_2+C_3=0,\\
&\tfrac{1}{6}C_0+\tfrac{1}{2}C_1=-\tfrac{2550499316977}{1671295626000},\\
f_2'(1)=0 \;\Rightarrow\; & \tfrac{2}{15925}-\tfrac{2}{2275}+\tfrac{1}{350}-\tfrac{17}{550}+\tfrac{101}{11340}+\tfrac{161543}{1590120}-\tfrac{15067}{35280}+\tfrac{9434}{11025}+\\
&+\tfrac{2486929}{952560}-\tfrac{3928307}{396900}+\tfrac{2610983}{297675}+\tfrac{1}{2}C_0+C_1+C_2=0,\\
&\tfrac{1}{2}C_0+C_1=-\tfrac{428661712069}{214880866200}.
\end{aligned}
\tag{4.17a}
$$

for suction :

$$
\begin{aligned}
f_2(1)=0 \;\Rightarrow\; & \tfrac{2}{238875}-\tfrac{1}{15925}+\tfrac{1}{4550}-\tfrac{17}{6600}+\tfrac{317}{124740}-\tfrac{20191}{2271600}-\tfrac{703}{317520}+\tfrac{2777}{44100}+\\
&+\tfrac{2486929}{6667920}-\tfrac{1941287}{2381400}+\tfrac{852338}{1488375}+\tfrac{1}{6}C_0+\tfrac{1}{2}C_1+C_2+C_3=0,\\
&\tfrac{1}{6}C_0+\tfrac{1}{2}C_1=-\tfrac{2744265969017}{15041660634000},\\
f_2'(1)=0 \;\Rightarrow\; & \tfrac{2}{15925}-\tfrac{2}{2275}+\tfrac{1}{350}-\tfrac{17}{550}+\tfrac{317}{11340}-\tfrac{20191}{227160}-\tfrac{703}{35280}+\tfrac{5554}{11025}+\\
&+\tfrac{2486929}{952560}-\tfrac{1941287}{396900}+\tfrac{852338}{297675}+\tfrac{1}{2}C_0+C_1+C_2=0,\\
&\tfrac{1}{2}C_0+C_1=-\tfrac{209955868999}{214880866200}.
\end{aligned}
\tag{4.17b}
$$

After trivial calculations we obtain

$$
\begin{array}{ll}
\text{For injection :} & \text{For suction :}\\
\tfrac{1}{6}C_0+\tfrac{1}{2}C_1=-\tfrac{2550499316977}{1671295626000} & \tfrac{1}{6}C_0+\tfrac{1}{2}C_1=-\tfrac{2744265969017}{15041660634000}\\
\tfrac{1}{2}C_0+C_1=-\tfrac{428661712069}{214880866200} & \tfrac{1}{2}C_0+C_1=-\tfrac{210050322127}{214880866200}\\
C_2=C_3=0 & C_2=C_3=0
\end{array}
\tag{4.18}
$$

and the coefficients C_0, C_1, C_2, C_3 for the both cases are reduced at the table,

$$
\begin{array}{ll}
\text{For injection :} & \text{For suction :}\\
C_0=+\tfrac{3975666965189}{626735859750} & C_0=-\tfrac{9208378891896}{2506943439000}\\
C_1=-\tfrac{38857161713549}{7520830317000} & C_1=+\tfrac{6464112922879}{7520830317000}\\
C_2=C_3=0 & C_2=C_3=0
\end{array}
\tag{4.19}
$$

for injection :

$$
\begin{aligned}
f_2=&\tfrac{2}{79625}\eta^{15}-\tfrac{1}{15925}\eta^{14}+\tfrac{1}{4550}\eta^{13}-\tfrac{17}{6600}\eta^{12}+\tfrac{101}{124740}\eta^{11}+\tfrac{10103}{15901200}\eta^{10}-\tfrac{15067}{317520}\eta^{9}+\\
&+\tfrac{4717}{44100}\eta^{8}+\tfrac{2486929}{6667920}\eta^{7}-\tfrac{2934797}{2381400}\eta^{6}+\tfrac{3463321}{1488375}\eta^{5}+\tfrac{3975666965189}{3760415158500}\eta^{3}-\tfrac{38857161713549}{15041660634000}\eta^{2};
\end{aligned}
\tag{4.20a}
$$

for suction :

$$
\begin{aligned}
f_2=&\tfrac{2}{238875}\eta^{15}-\tfrac{1}{15925}\eta^{14}+\tfrac{1}{4550}\eta^{13}-\tfrac{17}{6600}\eta^{12}+\tfrac{317}{124740}\eta^{11}-\tfrac{20191}{2271600}\eta^{10}-\tfrac{703}{317520}\eta^{9}+\\
&+\tfrac{2777}{44100}\eta^{8}+\tfrac{2486929}{6667920}\eta^{7}-\tfrac{1941287}{2381400}\eta^{6}+\tfrac{852338}{1488375}\eta^{5}-\tfrac{1534729815316}{2506943439000}\eta^{3}+\tfrac{6464112922879}{15041660634000}\eta^{2}.
\end{aligned}
\tag{4.20b}
$$

Finally, our main results can be reduced by few simple analytical expressions to obtain u and v taking into account the relationships (1.4) between dimensional and dimensionless quantities,

$$u^* = \partial\psi / \partial y = (x/h) f'_\eta(\eta),\ v^* = -\left(\partial\psi / dx\right) = -f(\eta),$$
$$u = u^* V = V \cdot (x/h) f'_\eta(\eta,\ v = v^* V = -V \cdot f(\eta);$$
$$f = f_0 + f_1 \cdot \mathrm{Re} + f_2 \cdot \mathrm{Re}^2 + O(\mathrm{Re}^3),\ f' = f'_0 + f'_1 \cdot \mathrm{Re} + f'_2 \cdot \mathrm{Re}^2 + O(\mathrm{Re}^3)$$

due to $\psi(x,\eta) = xf(\eta)$.

For injection: $\eta = y/h$ with $h = h_1$, and

$$f_0 = -2\eta^3 + 3\eta^2, \quad f'_0 = -6(\eta^2 - \eta), \tag{4.21a}$$
$$f_1 = \tfrac{2}{35}\eta^7 - \tfrac{1}{5}\eta^6 + \tfrac{3}{10}\eta^5 - \tfrac{1577}{378}\eta^3 + \tfrac{1861}{315}\eta^2, \quad f'_1 = \tfrac{2}{5}\eta^6 - \tfrac{6}{5}\eta^5 + \tfrac{3}{2}\eta^4 - \tfrac{1577}{126}\eta^2 + \tfrac{3722}{315}\eta,$$
$$f_2 = \tfrac{2}{79625}\eta^{15} - \tfrac{1}{15925}\eta^{14} + \tfrac{1}{4550}\eta^{13} - \tfrac{17}{6600}\eta^{12} + \tfrac{101}{124740}\eta^{11} + \tfrac{10103}{15901200}\eta^{10} - \tfrac{15067}{317520}\eta^9 + \tfrac{4717}{44100}\eta^8 +$$
$$+ \tfrac{2486929}{6667920}\eta^7 - \tfrac{2934797}{2381400}\eta^6 + \tfrac{3463321}{1488375}\eta^5 + \tfrac{3975666965189}{3760415158500}\eta^3 - \tfrac{38857161713549}{15041660634000}\eta^2,$$
$$f'_2 = \tfrac{2}{15925}\eta^{14} - \tfrac{2}{2275}\eta^{13} + \tfrac{1}{350}\eta^{12} - \tfrac{17}{550}\eta^{11} + \tfrac{101}{11340}\eta^{10} + \tfrac{161543}{1590120}\eta^9 - \tfrac{15067}{35280}\eta^8 + \tfrac{9434}{11025}\eta^7 +$$
$$+ \tfrac{2486929}{952560}\eta^6 - \tfrac{3928307}{396900}\eta^5 + \tfrac{2610983}{297675}\eta^4 + \tfrac{3975666965189}{1253471719500}\eta^2 - \tfrac{38857161713549}{7520830317000}\eta;$$

For suction: $\eta = y/h$ with $h = h_2$, and

$$f_0 = 2\eta^3 - 3\eta^2, \quad f'_0 = 6(\eta^2 - \eta),$$
$$f_1 = \tfrac{2}{35}\eta^7 - \tfrac{1}{5}\eta^6 + \tfrac{3}{10}\eta^5 - \tfrac{1577}{378}\eta^3 + \tfrac{1861}{315}\eta^2, \quad f'_1 = \tfrac{2}{5}\eta^6 - \tfrac{6}{5}\eta^5 + \tfrac{3}{2}\eta^4 - \tfrac{1577}{126}\eta^2 + \tfrac{3722}{315}\eta,$$
$$f_2 = \tfrac{2}{238875}\eta^{15} - \tfrac{1}{15925}\eta^{14} + \tfrac{1}{4550}\eta^{13} - \tfrac{17}{6600}\eta^{12} + \tfrac{317}{124740}\eta^{11} - \tfrac{20191}{2271600}\eta^{10} - \tfrac{703}{317520}\eta^9 + \tfrac{2777}{44100}\eta^8 + \tag{4.21b}$$
$$+ \tfrac{2486929}{6667920}\eta^7 - \tfrac{1941287}{2381400}\eta^6 + \tfrac{852338}{1488375}\eta^5 - \tfrac{1534729815316}{2506943439000}\eta^3 + \tfrac{6464112922879}{15041660634000}\eta^2,$$
$$f'_2 = \tfrac{2}{15925}\eta^{14} - \tfrac{2}{2275}\eta^{13} + \tfrac{1}{350}\eta^{12} - \tfrac{17}{550}\eta^{11} + \tfrac{317}{11340}\eta^{10} - \tfrac{20191}{227160}\eta^9 - \tfrac{703}{35280}\eta^8 + \tfrac{5554}{11025}\eta^7 +$$
$$+ \tfrac{2486929}{952560}\eta^6 - \tfrac{1941287}{396900}\eta^5 + \tfrac{852338}{297675}\eta^4 - \tfrac{4604189445948}{2506943439000}\eta^2 + \tfrac{6464112922879}{7520830317000}\eta\ ,$$

Common Algorithm

Follow our scheme (Fig. 1) we have two chamber with the initial pressures P_1 and P_2 with $P_1 > P_2$. Hence, calculation formulas (4.21) are valid for the both champers with may be different values $h = h_1$ and h_2. Moreover, for the right chamber, the porous wall is the ***injection boundary***, and for the left chamber the porous wall is the ***suction boundary***.

Firstly we should find the pressure drop $\Delta P = P_1 - P_2$ and the boundary velocity at the porous wall $V = \frac{k}{\mu}\frac{\Delta P}{\delta}$, following (1.3). There should be used dimensional values P_1, P_2.

The second step: using this value V we can calculate the Reynolds number Re and the function f with all its derivatives under formulas (4.21).

The third step is to find new ΔP from (3.5),

$$\Delta P = \tfrac{1}{2}(\delta / h)^2 A\ , \ A = \mathrm{Re}(f'^2_\eta - ff''_{\eta\eta}) - f'''_{\eta\eta\eta}$$

Calculated using the expressions for

$$f = f_0 + \mathrm{Re}\cdot f_1 + \mathrm{Re}^2\cdot f_2, \quad f' = f'_0 + \mathrm{Re}\cdot f'_1 + \mathrm{Re}^2\cdot f'_2\ ,$$
$$f'' = f''_0 + \mathrm{Re}\cdot f''_1 + \mathrm{Re}^2\cdot f''_2, \quad f''' = f'''_0 + \mathrm{Re}\cdot f'''_1 + \mathrm{Re}^2\cdot f'''_2 \tag{5.1}$$

Defined by the formulas (4.8), (4.12). and (4.18a). Note, that to find ΔP it is necessary to obtain $f_0, f_1, f_2, f_0', f_1', f_2', f_0'', f_1'', f_2'', f_0''', f_1''', f_2'''$, not only $f_0, f_1, f_2, f_0', f_1', f_2'$ as it was necessary for u and v, see (4.21) For injection the coefficients C_0, C_1 taken from the left hand side column of the table (4.19) to find the corresponding terms under the expressions,

$$
\begin{aligned}
&f_0 = -(2\eta^3 - 3\eta^2),\ f_0' = -6(\eta^2 - \eta),\ f_0'' = -6(2\eta - 1),\ f_0''' = -12;\\
&f_1' = \tfrac{2}{5}\eta^6 - \tfrac{6}{5}\eta^5 + \tfrac{3}{2}\eta^4 - \tfrac{1577}{126}\eta^2 + \tfrac{3722}{315}\eta,\\
&f_1'' = \tfrac{12}{5}\eta^5 - 6\eta^4 + 6\eta^3 - \tfrac{1577}{63}\eta + \tfrac{3722}{315},\\
&f_1''' = 12\eta^4 - 24\eta^3 + 18\eta^2 - \tfrac{1577}{63};\\
&f_2 = \tfrac{2}{79625}\eta^{15} - \tfrac{1}{15925}\eta^{14} + \tfrac{1}{4550}\eta^{13} - \tfrac{17}{6600}\eta^{12} + \tfrac{101}{124740}\eta^{11} + \tfrac{10103}{15901200}\eta^{10} - \tfrac{15067}{317520}\eta^9 + \tfrac{4717}{44100}\eta^8 +\\
&\quad + \tfrac{2486929}{6667920}\eta^7 - \tfrac{2934797}{2381400}\eta^6 + \tfrac{3463321}{1488375}\eta^5 + \tfrac{3975666965189}{3760415158500}\eta^3 - \tfrac{38857161713549}{15041660634000}\eta^2,\\
&f_2' = \tfrac{2}{15925}\eta^{14} - \tfrac{2}{2275}\eta^{13} + \tfrac{1}{350}\eta^{12} - \tfrac{17}{550}\eta^{11} + \tfrac{101}{11340}\eta^{10} + \tfrac{161543}{1590120}\eta^9 - \tfrac{15067}{35280}\eta^8 + \tfrac{9434}{11025}\eta^7 +\\
&\quad + \tfrac{2486929}{952560}\eta^6 - \tfrac{3928307}{396900}\eta^5 + \tfrac{2610983}{297675}\eta^4 + \tfrac{3975666965189}{1253471719500}\eta^2 - \tfrac{38857161713549}{7520830317000}\eta\ ,\\
&f_2'' = \tfrac{4}{2275}\eta^{13} - \tfrac{12}{175}\eta^{12} + \tfrac{6}{175}\eta^{11} - \tfrac{17}{50}\eta^{10} + \tfrac{101}{1134}\eta^9 + \tfrac{161543}{176680}\eta^8 - \tfrac{15067}{4410}\eta^7 + \tfrac{9434}{1575}\eta^6 +\\
&\quad + \tfrac{2486929}{158760}\eta^5 - \tfrac{3928307}{79380}\eta^4 + \tfrac{10443932}{297675}\eta^3 + \tfrac{3975666965189}{626735859750}\eta - \tfrac{38857161713549}{7520830317000},\\
&f_2''' = \tfrac{4}{175}\eta^{12} - \tfrac{24}{175}\eta^{11} + \tfrac{66}{175}\eta^{10} - \tfrac{17}{5}\eta^9 + \tfrac{101}{126}\eta^8 + \tfrac{161543}{22085}\eta^7 - \tfrac{15067}{630}\eta^6 + \tfrac{18868}{525}\eta^5 +\\
&\quad + \tfrac{2486929}{31752}\eta^4 - \tfrac{3928307}{19845}\eta^3 + \tfrac{10443932}{99225}\eta^2 + \tfrac{3975666965189}{626735859750}.
\end{aligned}
\tag{5.2a}
$$

For suction we can simultaneously use the expressions (4.8), (4.12), and (4.18b) with the coefficients C_0, C_1 taken from the right hand side column of the table (4.19) to find the corresponding terms $f_0, f_1, f_2, f_0', f_1', f_2', f_0'', f_1'', f_2'', f_0''', f_1''', f_2'''$ under the expressions,

$$
\begin{aligned}
&f_0 = (2\eta^3 - 3\eta^2),\ f_0' = 6(\eta^2 - \eta),\ f_0'' = 6(2\eta - 1),\ f_0''' = 12,\\
&f_1' = \tfrac{2}{5}\eta^6 - \tfrac{6}{5}\eta^5 + \tfrac{3}{2}\eta^4 - \tfrac{1577}{126}\eta^2 + \tfrac{3722}{315}\eta,\\
&f_1'' = \tfrac{12}{5}\eta^5 - 6\eta^4 + 6\eta^3 - \tfrac{1577}{63}\eta + \tfrac{3722}{315},\\
&f_1''' = 12\eta^4 - 24\eta^3 + 18\eta^2 - \tfrac{1577}{63};\\
&f_2 = \tfrac{2}{238875}\eta^{15} - \tfrac{1}{15925}\eta^{14} + \tfrac{1}{4550}\eta^{13} - \tfrac{17}{6600}\eta^{12} + \tfrac{317}{124740}\eta^{11} - \tfrac{20191}{2271600}\eta^{10} - \tfrac{703}{317520}\eta^9 + \tfrac{2777}{44100}\eta^8 +\\
&\quad + \tfrac{2486929}{6667920}\eta^7 - \tfrac{1941287}{2381400}\eta^6 + \tfrac{852338}{1488375}\eta^5 - \tfrac{1534729815316}{2506943439000}\eta^3 + \tfrac{6464112922879}{15041660634000}\eta^2,
\end{aligned}
\tag{5.2b}
$$

$$
\begin{aligned}
&f_2' = \tfrac{2}{15925}\eta^{14} - \tfrac{2}{2275}\eta^{13} + \tfrac{1}{350}\eta^{12} - \tfrac{17}{550}\eta^{11} + \tfrac{317}{11340}\eta^{10} - \tfrac{20191}{227160}\eta^9 - \tfrac{703}{35280}\eta^8 + \tfrac{5554}{11025}\eta^7 +\\
&\quad + \tfrac{2486929}{952560}\eta^6 - \tfrac{1941287}{396900}\eta^5 + \tfrac{852338}{297675}\eta^4 - \tfrac{4604189445948}{2506943439000}\eta^2 + \tfrac{6464112922879}{7520830317000}\eta\ ,\\
&f_2'' = \tfrac{4}{2275}\eta^{13} - \tfrac{2}{175}\eta^{12} + \tfrac{6}{175}\eta^{11} - \tfrac{17}{50}\eta^{10} + \tfrac{317}{1134}\eta^9 - \tfrac{20191}{25240}\eta^8 - \tfrac{703}{4410}\eta^7 + \tfrac{5554}{1575}\eta^6 +\\
&\quad + \tfrac{2486929}{158760}\eta^5 - \tfrac{1941287}{79380}\eta^4 + \tfrac{3409352}{297675}\eta^3 - \tfrac{9208378891896}{2506943439000}\eta + \tfrac{6464112922879}{7520830317000},\\
&f_2''' = \tfrac{4}{175}\eta^{12} - \tfrac{24}{175}\eta^{11} + \tfrac{66}{175}\eta^{10} - \tfrac{17}{5}\eta^9 + \tfrac{317}{126}\eta^8 - \tfrac{20191}{3155}\eta^7 - \tfrac{703}{630}\eta^6 + \tfrac{11108}{525}\eta^5 +\\
&\quad + \tfrac{2486929}{31752}\eta^4 - \tfrac{1941287}{19845}\eta^3 + \tfrac{3409352}{99225}\eta^2 - \tfrac{9208378891896}{2506943439000}.
\end{aligned}
\tag{5.2b}
$$

Substituting $\eta = 1$ in (5.2a) – (5.2b) one obtains for injection,

$$
\begin{aligned}
&f_0 = 1, \quad f_0' = 0, \quad f_0'' = -6, \quad f_0''' = -12 \ ; \\
&f_1' = \tfrac{2}{5} - \tfrac{6}{5} + \tfrac{3}{2} - \tfrac{1577}{126} + \tfrac{3722}{315} = 0, \ f_1'' = \tfrac{12}{5} - \tfrac{1577}{63} + \tfrac{3722}{315} = \tfrac{3407}{315}, \ f_1''' = 6 - \tfrac{1577}{63} = \tfrac{1199}{63}; \\
&f_2 = \tfrac{2}{79625} - \tfrac{1}{15925} + \tfrac{1}{4550} - \tfrac{17}{6600} + \tfrac{101}{124740} + \tfrac{10103}{15901200} - \tfrac{15067}{317520} + \tfrac{4717}{44100} + \tfrac{2486929}{6667920} - \tfrac{2934797}{2381400} + \\
&\quad + \tfrac{3463321}{1488375} + \tfrac{3975666965189}{3760415158500} - \tfrac{38857161713549}{15041660634000} = 0, \\
&f_2' = \tfrac{2}{15925} - \tfrac{2}{2275} + \tfrac{1}{350} - \tfrac{17}{550} + \tfrac{101}{11340} + \tfrac{161543}{1590120} - \tfrac{15067}{35280} + \tfrac{9434}{11025} + \tfrac{2486929}{952560} - \tfrac{3928307}{396900} + \tfrac{2610983}{297675} + \\
&\quad + \tfrac{3975666965189}{1253471719500} - \tfrac{38857161713549}{7520830317000} = 0, \\
&f_2'' = \tfrac{4}{2275} - \tfrac{12}{175} + \tfrac{6}{175} - \tfrac{17}{50} + \tfrac{101}{1134} + \tfrac{161543}{176680} - \tfrac{15067}{4410} + \tfrac{9434}{1575} + \tfrac{2486929}{158760} - \tfrac{3928307}{79380} + \tfrac{10443932}{297675} + \\
&\quad + \tfrac{3975666965189}{626735859750} - \tfrac{38857161713549}{7520830317000} = 5\,\tfrac{4838552565389}{7520830317000} = \tfrac{42442704150389}{7520830317000}, \\
&f_2''' = \tfrac{4}{175} - \tfrac{24}{175} + \tfrac{66}{175} - \tfrac{17}{5} + \tfrac{101}{126} + \tfrac{161543}{22085} - \tfrac{15067}{630} + \tfrac{18868}{525} + \tfrac{2486929}{31752} - \tfrac{3928307}{19845} + \tfrac{10443932}{99225} + \\
&\quad + \tfrac{3975666965189}{626735859750} = 8\,\tfrac{2443762716071}{2506943439000} = \tfrac{22499310228071}{2506943439000},
\end{aligned}
\tag{5.3a}
$$

and for suction,

$$
\begin{aligned}
&f_0 = -1, \quad f_0' = 0, \quad f_0'' = 6, \quad f_0''' = 12 \ ; \\
&f_1' = \tfrac{2}{5} - \tfrac{6}{5} + \tfrac{3}{2} - \tfrac{1577}{126} + \tfrac{3722}{315} = 0, \ f_1'' = \tfrac{12}{5} - \tfrac{1577}{63} + \tfrac{3722}{315} = \tfrac{3407}{315}, \ f_1''' = 6 - \tfrac{1577}{63} = \tfrac{1199}{63}; \\
&f_2 = \tfrac{2}{238875} - \tfrac{1}{15925} + \tfrac{1}{4550} - \tfrac{17}{6600} + \tfrac{317}{124740} - \tfrac{20191}{2271600} - \tfrac{703}{317520} + \tfrac{2777}{44100} + \tfrac{2486929}{6667920} - \tfrac{1941287}{2381400} + \\
&\quad + \tfrac{852338}{1488375} - \tfrac{1534729815316}{2506943439000} + \tfrac{6464112922879}{15041660634000} = 0 \ , \\
&f_2' = \tfrac{2}{15925} - \tfrac{2}{2275} + \tfrac{1}{350} - \tfrac{17}{550} + \tfrac{317}{11340} - \tfrac{20191}{227160} - \tfrac{703}{35280} + \tfrac{5554}{11025} + \tfrac{2486929}{952560} - \tfrac{1941287}{396900} + \tfrac{852338}{297675} - \\
&\quad - \tfrac{4604189445948}{2506943439000} + \tfrac{6464112922879}{7520830317000} = 0 \ , \\
&f_2'' = \tfrac{4}{2275} - \tfrac{2}{175} + \tfrac{6}{175} - \tfrac{17}{50} + \tfrac{317}{1134} - \tfrac{20191}{25240} - \tfrac{703}{4410} + \tfrac{5554}{1575} + \tfrac{2486929}{158760} - \tfrac{1941287}{79380} + \tfrac{3409352}{297675} - \\
&\quad - \tfrac{9208378891896}{2506943439000} + \tfrac{6464112922879}{7520830317000} = 2\,\tfrac{2856702157061}{7520830317000} = \tfrac{17898362791061}{7520830317000}, \\
&f_2''' = \tfrac{4}{175} - \tfrac{24}{175} + \tfrac{66}{175} - \tfrac{17}{5} + \tfrac{317}{126} - \tfrac{20191}{3155} - \tfrac{703}{630} + \tfrac{11108}{525} + \tfrac{2486929}{31752} - \tfrac{1941287}{19845} + \tfrac{3409352}{99225} - \\
&\quad - \tfrac{9208378891896}{2506943439000} = 24\,\tfrac{174644137273}{835647813000} = \tfrac{20230191649273}{835647813000}.
\end{aligned}
\tag{5.3b}
$$

As a result we have for injection with Reynolds number $\mathrm{Re} = \rho h_1 V \ / \ \mu$,

$$
\begin{aligned}
&f = 1 + 0 \cdot \mathrm{Re} + 0 \cdot \mathrm{Re}^2 = 1, \quad f' = 0 + 0 \cdot \mathrm{Re} + 0 \cdot \mathrm{Re}^2 = 0, \\
&f'' = -6 + \tfrac{3407}{315}\mathrm{Re} + \tfrac{42442704150389}{7520830317000}\mathrm{Re}^2, \quad f''' = -12 + \tfrac{1199}{63}\mathrm{Re} + \tfrac{22499310228071}{2506943439000}\mathrm{Re}^2,
\end{aligned}
\tag{5.4a}
$$

and for suction, with the Reynolds number $\mathrm{Re} = \rho h_2 V \ / \ \mu$

$$
\begin{aligned}
&f = -1 + 0 \cdot \mathrm{Re} + 0 \cdot \mathrm{Re}^2 = -1, \quad f' = 0 + 0 \cdot \mathrm{Re} + 0 \cdot \mathrm{Re}^2 = 0, \\
&f'' = 6 + \tfrac{3407}{315}\mathrm{Re} + \tfrac{17898362791061}{7520830317000}\mathrm{Re}^2, \quad f''' = 12 + \tfrac{1199}{63}\mathrm{Re} + \tfrac{20230191649273}{835647813000}\mathrm{Re}^2 .
\end{aligned}
\tag{5.4b}
$$

Finally, we obtained formulas for $\Delta P_{\text{injection}}$, for injection, with the Reynolds number $\mathrm{Re}_1 = \frac{\rho h_1 V}{\mu}$:

$$\begin{aligned}
A &= \mathrm{Re}_1(f_\eta'^2 - ff_{\eta\eta}'') - f_{\eta\eta\eta}''' = \\
&= \mathrm{Re}_1\left(6 - \tfrac{3407}{315}\mathrm{Re}_1 - \tfrac{42442704150389}{7520830317000}\mathrm{Re}_1^2\right) + \left(12 - \tfrac{1199}{63}\mathrm{Re}_1 - \tfrac{22499310228071}{2506943439000}\mathrm{Re}_1^2\right) = \\
&= 12 + 6\,\mathrm{Re}_1 - \tfrac{1199}{63}\mathrm{Re}_1 - \tfrac{3407}{315}\mathrm{Re}_1^2 - \tfrac{22499310228071}{2506943439000}\mathrm{Re}_1^2 = \\
&= 12 - \tfrac{821}{63}\cdot\mathrm{Re}_1 - 19\tfrac{1982166781271}{2506943439000}\cdot\mathrm{Re}_1^2 = 12 - \tfrac{821}{63}\cdot\mathrm{Re}_1 - \tfrac{49614092122271}{2506943439000}\cdot\mathrm{Re}_1^2,
\end{aligned}$$

$$\Delta P_{\text{injection}} = \tfrac{1}{2}(x/h_1)^2\left(12 - \tfrac{821}{63}\cdot\mathrm{Re}_1 - \tfrac{49614092122271}{2506943439000}\cdot\mathrm{Re}_1^2\right), \tag{5.5a}$$

and $\Delta P_{\text{suction}}$, for suction, with the Reynolds number $\mathrm{Re}_2 = \frac{\rho h_2 V}{\mu}$:

$$\begin{aligned}
A &= \mathrm{Re}_2(f_\eta'^2 - ff_{\eta\eta}'') - f_{\eta\eta\eta}''' = \\
&= -12 + 6\,\mathrm{Re}_2 + \tfrac{1199}{63}\mathrm{Re}_2 + \tfrac{3407}{315}\mathrm{Re}_2^2 + \tfrac{20230191649273}{835647813000}\mathrm{Re}_2^2 = \\
&= -12 + \tfrac{1577}{63}\mathrm{Re}_2 + 35\tfrac{20778825673}{835647813000}\mathrm{Re}_2^2 = -12 + \tfrac{1577}{63}\mathrm{Re}_2 + \tfrac{29268452280673}{835647813000}\mathrm{Re}_2^2,
\end{aligned}$$

$$\Delta P_{\text{suction}} = \tfrac{1}{2}(x/h_2)^2\left(-12 + \tfrac{1577}{63}\mathrm{Re}_2 + \tfrac{29268452280673}{835647813000}\mathrm{Re}_2^2\right). \tag{5.5b}$$

The last two expressions define the new pressures at the both chambers

$$P_1^{(1)} = P_1^{(0)} - \left|\Delta P_{\text{injection}}\right|, \quad P_2^{(1)} = P_2^{(0)} + \left|\Delta P_{\text{suction}}\right|, \tag{5.6}$$

and the new pressure drop $\Delta P^{(1)} = P_1^{(1)} - P_2^{(1)}$ and the new velocity $V^{(1)}$,

$$\begin{aligned}
\Delta P^{(0)} &= P_1^{(0)} - P_2^{(0)}, \\
\Delta P^{(1)} &= \left(P_1^{(0)} - \left|\Delta P_{\text{injection}}\right|\right) - \left(P_2^{(0)} + \left|\Delta P_{\text{suction}}\right|\right) = \\
&\qquad = \Delta P^{(0)} - \tfrac{1}{2}(x/h_1)^2\left(\left|\Delta P_{\text{injection}}\right| + (h_1/h_2)^2\left|\Delta P_{\text{suction}}\right|\right), \\
\Delta P_{\text{injection}} &= 12 - \tfrac{821}{63}\cdot\mathrm{Re}_1 - \tfrac{49614092122271}{2506943439000}\cdot\mathrm{Re}_1^2, \quad \Delta P_{\text{suction}} = -12 + \tfrac{1577}{63}\mathrm{Re}_2 + \tfrac{29268452280673}{835647813000}\mathrm{Re}_2^2, \\
V^{(1)} &= (k/\mu)(\Delta P^{(1)}_{\text{dimensional}}/\delta) = (k/\mu)\cdot(\mu V^{(0)}/h_1)(\Delta P^{(1)}/\delta) = \\
&= (k/h_1\delta)\cdot V^{(0)}\left[\Delta P^{(0)} - \tfrac{1}{2}(x/h_1)^2\left(\left|\Delta P_{\text{injection}}\right| + (h_1/h_2)^2\left|\Delta P_{\text{suction}}\right|\right)\right].
\end{aligned} \tag{5.7}$$

In the partial case, when the both channels have the same width, i.e. if $h_1 = h_2 = h$ then,

$$\begin{aligned}
\Delta P^{(1)} &= \Delta P^{(0)} - \tfrac{1}{2}(x/h)^2\left(\tfrac{84}{7}\mathrm{Re} + \tfrac{2161606561}{141891750}\mathrm{Re}^2\right), \\
V^{(1)} &= (k/h\delta)\cdot V^{(0)}\left[\Delta P^{(0)} - \tfrac{1}{2}(x/h)^2\left(\tfrac{84}{7}\mathrm{Re} + \tfrac{2161606561}{141891750}\mathrm{Re}^2\right)\right].
\end{aligned} \tag{5.7a}$$

In Section 5, all pressures are dimensionless, due to they were namely calculated this way. The only exception is the value $\Delta P^{(1)}_{\text{dimensional}}$ used to calculate dimensional value $V^{(1)}$; this pressure drop is designated as *dimensional*.

It can be watched that the value A and, consequently, ΔP and V, at each step, depend only on Reynolds number Re because all terms $f_0, f_1, f_2, f_0', f_1', f_2', f_0'', f_1'', f_2'', f_0''', f_1''', f_2'''$ calculated using (5.3a) – (5.3b) are constants. Note, that the velocity is defined by real value of the pressure drop ΔP, and consequently, by the both dimensional pressures P_1, P_2 to be found after multiplying the corresponding values above (5.6) by the multipliers $\mu V / h_1$ and $\mu V / h_2$, follow to the inverse dimensionless transformation (1.4a), with the choice h_1, h_2 in accordance with the considered chamber.

At the forth step we should compare our initial value of velocity $V = (k / \mu)(P_1 - P_2) / \delta$ calculated for initial pressure drop with the velocity calculated for the new pressure drop found at the previous step. If $\left|V^{(1)} - V^{(0)}\right| / \min(V^{(1)}, V^{(0)}) < \varepsilon$, where ε is accuracy sufficient for the study, then the calculation by this algorithm can be considered complete. If not, then we need to return to the 2nd step with the new value V there it is necessary to use dimensional pressures P_1 and P_2 using the transform (1.4a).

Note, that for the forth step (**d**) it is more convenient to use dimensionless values of pressures P_1 and P_2 (see Eq. (1.4a))**1/** instead of dimensional velocity values $V^{(0)}$ and $V^{(1)}$ due to the ratio

$$\left|V^{(1)} - V^{(0)}\right| / \min\left\{V^{(1)}, V^{(0)}\right\} < \varepsilon \tag{5.8}$$

is a dimensionless value and so the inequality (5.8) may be represented as,

$$\frac{\left|V^{(1)} - V^{(0)}\right|}{\min(V^{(1)}, V^{(0)})} = \frac{\left|\left(P_1^{(1)} - \frac{h_1}{h_2} P_2^{(1)}\right) - \frac{V^{(0)}}{V^{(1)}} \cdot \left(P_1^{(0)} - \frac{h_1}{h_2} P_2^{(0)}\right)\right|}{\min\left\{\left(P_1^{(1)} - \frac{h_1}{h_2} P_2^{(1)}\right), \frac{V^{(0)}}{V^{(1)}} \cdot \left(P_1^{(0)} - \frac{h_1}{h_2} P_2^{(0)}\right)\right\}} < \varepsilon \tag{5.9}$$

At the final fifth step, after satisfying inequality (5.9), the velocity field $\mathbf{u} = (u, v)$ for the both chambers can be calculated using expressions (4.21a,b). The general block-scheme of the calculation algorithm is presented in the Appendix.

Common Algorithm for two components flow

Let us consider the case when the water vapor moves in the air co-flow when the vapor is a small component compared to air flow, i.e. when $\dot{m} << \dot{m}_{Air}$, where

$$\dot{m}_{vapor} = \dot{M}_{vapor} / (\dot{M}_{Air} + \dot{M}_{vapor}), \ \dot{m}_{Air} = 1 - \dot{m}_{vapor}, \tag{6.1}$$

and $\dot{M}_{vapor}$, $\dot{M}_{vapor}$ are the mass flow of the vapor and air, correspondingly. In any case the air flow is the main one and the steam moves due to the Stokes forces.

1/ Recall that at the end of Section 1 we agreed that for simplicity, we omitted the asterisks (*) at the superscripts.

Naturally, in the case $\dot{m} << \dot{m}_{Air}$ the horizontal component of the vapor phase velocity v_{vapor} coincides with the velocity of air flow v_{Air}. Since both walls are impermeable for the air component, the air flow is a Poiseuille flow, with the parabolic profile of the velocity v_{Air},

$$v_{Air} = \frac{1}{2\mu} \frac{\partial P}{\partial x}(hy - y^2) \,. \tag{6.1}$$

It can be seen that the velocity v_{Air} satisfy to the boundary conditions, $v_{Air}(0) = v_{Air}(h) = 0$, and the maximal velocity value

$$v_{Air}^{\max} = -\frac{1}{8\mu} h^2 \frac{\partial P}{\partial x} \tag{6.2}$$

is reached in the center of the channel. Namely we should use the value

$$v = -\dot{m}_{vapor} V \cdot f(\eta) - \dot{m}_{Air} \frac{1}{2\mu} \frac{\partial P}{\partial x} (hy - y^2) =$$
$$= -\dot{m}_{vapor} V \cdot f(\eta) + \dot{m}_{Air} \frac{1}{2\mu} \left\{ \frac{\Delta P}{L} \mp \left[(f_\eta'^2 - ff_{\eta\eta}'') - f_{\eta\eta\eta}''' \right] \right\} x(hy - y^2) \tag{6.3}$$

instead of the value $v = v^* V = -V \cdot f(\eta)$ presented in (4.21). The signs, "$-$" and "$+$", correspond to the 1st (injection) and the 2nd (suction) channels, accordingly. The function $f(\eta)$ and its derivatives may be found from (4.3) as follows,

$$f^{(n)}(\eta) \approx f_0^{(n)} + f_1^{(n)} \cdot \mathrm{Re} + f_2^{(n)} \cdot \mathrm{Re}^2 , \tag{6.4}$$

Where, n is the order of the derivative order, and the coefficients $f_k^{(n)}$ $(k = 0, 1, 2)$ are presented in (5.2a) and (5.2b) for the both channels. Formula (6.3) is acceptable for common the case, i.e. when the vapor mass component $\dot{m}_{vapor}$ is not sufficiently low compared to the air co-flow mass component.

For the case when the vapor is a small component compared to air flow, i.e. when $\dot{m} << \dot{m}_{Air}$, we may neglect the first term, $-\dot{m}_{vapor} V \cdot f(\eta)$, in (6.2) and the pressure drop

$$-\frac{\partial P}{dx} = \mp x \left[(f_\eta'^2 - ff_{\eta\eta}'') - f_{\eta\eta\eta}''' \right] \tag{6.5}$$

in the second term due to the flow through a porous wall may be also neglected. Here, again, signs, "$-$" and "$+$", correspond to the 1st (injection) and the 2nd (suction) channels, accordingly. As a result, one may use the simplified formula,

$$v \approx \frac{1}{2\mu} \frac{\Delta P}{L} x(hy - y^2) . \tag{6.5}$$

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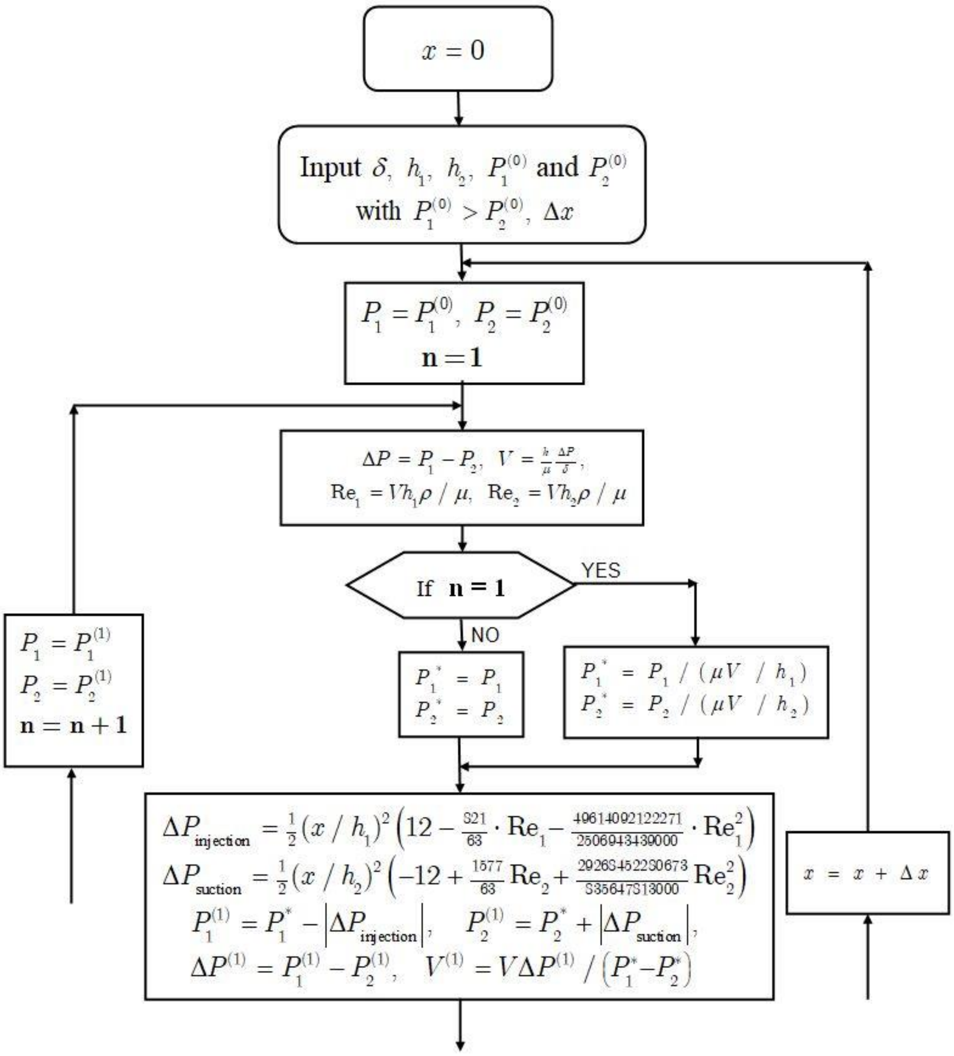
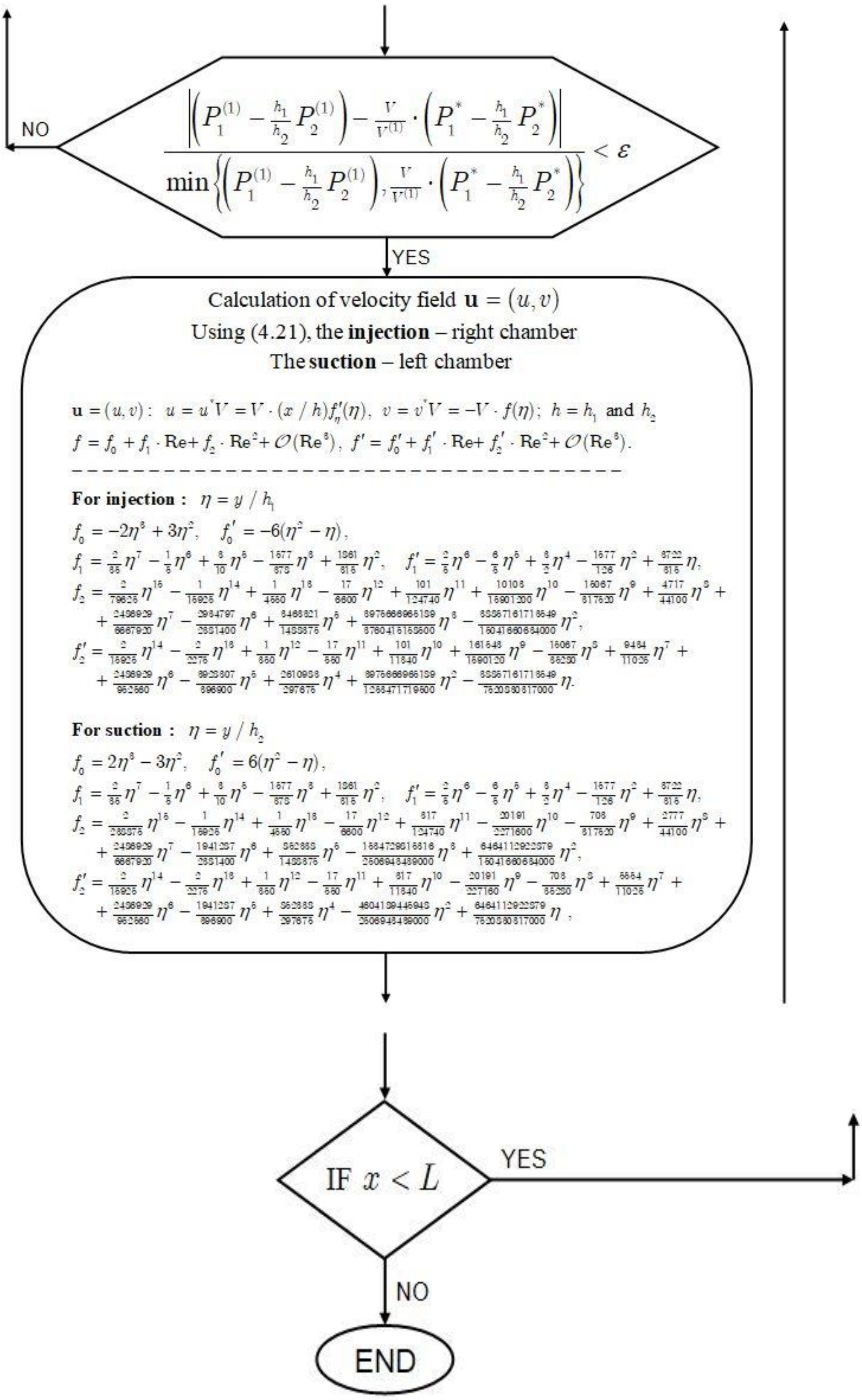
Appendix: Algorithm of the solution


$x = 0$
Input δ, h_1, h_2, $P_1^{(0)}$ and $P_2^{(0)}$ with $P_1^{(0)} > P_2^{(0)}$, Δx
$P_1 = P_1^{(0)}$, $P_2 = P_2^{(0)}$
$\mathbf{n = 1}$
$\Delta P = P_1 - P_2$, $V = \frac{h}{\mu}\frac{\Delta P}{\delta}$,
$\mathrm{Re}_1 = Vh_1\rho / \mu$, $\mathrm{Re}_2 = Vh_2\rho / \mu$
If $\mathbf{n = 1}$
YES
NO
$P_1^* = P_1$
$P_2^* = P_2$
$P_1^* = P_1 / (\mu V / h_1)$
$P_2^* = P_2 / (\mu V / h_2)$
$P_1 = P_1^{(1)}$
$P_2 = P_2^{(1)}$
$\mathbf{n = n + 1}$
$\Delta P_{\text{injection}} = \frac{1}{2}(x / h_1)^2\left(12 - \frac{821}{63}\cdot \mathrm{Re}_1 - \frac{49614092122271}{2506943439000}\cdot \mathrm{Re}_1^2\right)$
$\Delta P_{\text{suction}} = \frac{1}{2}(x / h_2)^2\left(-12 + \frac{1577}{63}\mathrm{Re}_2 + \frac{29268452280673}{835647813000}\mathrm{Re}_2^2\right)$
$P_1^{(1)} = P_1^* - \left|\Delta P_{\text{injection}}\right|$, $P_2^{(1)} = P_2^* + \left|\Delta P_{\text{suction}}\right|$,
$\Delta P^{(1)} = P_1^{(1)} - P_2^{(1)}$, $V^{(1)} = V\Delta P^{(1)} / \left(P_1^* - P_2^*\right)$
$x = x + \Delta x$



This procedure should be provided for all discrete length values $x \in [0, L]$, where L is the length of the chambers (see Fig. 1).