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REVIEW ARTICLE

ON THE ANALYSIS OF SOME THEORETIC GRAPH PROPERTIES OF THE INTERSECTION GRAPH OF Zn

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ABSTRACT

Abstract. This study investigates the structural properties of the Intersection Graph $\Gamma_{int}(Z_n)$ of the subgroups of Z_n, focusing on cases where n = p, n = pq, n = p^2 (p, q primes), and n = 2^k (k a natural number). The research examines the connectedness of the graph. The results reveal unique properties of $\Gamma_{int}(Z_n)$ for each case of connectedness. Notably, for n = 2^k, the graph is regular, complete, and exhibits rapid growth in size as k increases. This study provides a comprehensive understanding of the structural properties of $\Gamma_{int}(Z_n)$ and contributes to the existing body of knowledge on graph theory and group theory.

Keywords: Univalent function, radius of star like and radius of convexity.

INTRODUCTION

The importance of graph theory as a field of study stems from the fact that it is a powerful tool for modeling and analyzing a number of algebraic theoretic concepts such as optimization processing and combinatorial problems, including in theoretical computer science. Graph theory finds applications in various real-world scenarios. Some recent references showcasing the application of graph theory include:

- **Social Networks**: Analyzing social networks using graph theory helps in understanding relationships between individuals, identifying influencers, and predicting trends [6].
- **Bioinformatics**: Graph theory is used in bioinformatics for analyzing biological networks like protein-protein interactions or gene regulatory networks [3].
- **Transportation Networks**: Modeling transportation systems as graphs aids in optimizing routes, traffic flow analysis, and infrastructure planning [2].
- **Subgroups of Z_n**: In the study of subgroups of Z_n using the graph theoretic properties of intersection graphs, researchers analyze the structure and properties of subgroups within the additive group Z_n. Subgroups are subsets of a group that form a group under the same operation as the original group. By representing these subgroups as vertices in an intersection graph and connecting them based on their intersections, researchers can gain insights into the relationships between different subgroups. An intersection graph is a type of graph that represents intersections between sets. In the context of subgroup analysis in group theory, intersection graphs play a crucial role in understanding the relationships between subgroups. Specifically, when studying subgroups of Z_n (the additive group of integers modulo n), the intersection graph can be used to visualize how these subgroups intersect with each other.

• **The intersection graph of Z_n**: This is a graph whose vertices correspond to the elements of Z_n, and two vertices are subgroups in a cyclic group [1]. This paper explores the intersection graphs of subgroups in cyclic groups like Z_n and investigates their structural properties using advanced graph theoretical techniques.

METHODOLOGY

This section provides an explanation of the methodology employed to find the properties of the intersection graph. The theoretic properties to be investigated are as follows:

(1) Vertex degree
(2) Completeness
(3) Connectedness
(4) Size of a graph
(5) Independence number
(6) Chromatic number
(7) Clique number

Subgroups of Z_n. Theorem 1 (Subgroup Theorem)[4]: For a cyclic group Z_n (where n is a positive integer), every subgroup is also cyclic, and for each positive divisor k of n, there is a unique subgroup of order k, which is generated by $\left\langle \frac{n}{k} \right\rangle$.
It means for each factor of n, we have a unique subgroup. Therefore, assuming n has k + 1 factors, then Z_n will have k + 1 subgroups, including k, non-trivial subgroups.

For instance, if n = pq has a set of factors given by {1, p, q, n} where p and q are prime numbers, then we may have the following subgroups:

• Subgroup 1: $\left\langle \frac{n}{1} \right\rangle = \langle n \rangle = \{0\}$
• Subgroup 2: $\left\langle \frac{n}{p} \right\rangle = \langle q \rangle$
• Subgroup 3: $\left\langle \frac{n}{q} \right\rangle = \langle p \rangle$
• Subgroup 4: $\left\langle \frac{n}{n} \right\rangle = \langle 1 \rangle = Zn$

Example 2.1

Let Z16 = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}.

The factors of 16 are: k = {1, 2, 4, 8, 16}.

The corresponding subgroups are:

$\left\langle \frac{16}{1} \right\rangle = \langle 16 \rangle = \{0\}$
$\left\langle \frac{16}{2} \right\rangle = \langle 8 \rangle = \{0, 8\}$
$\left\langle \frac{16}{4} \right\rangle = \langle 4 \rangle = \{0, 4, 8, 12\}$
$\left\langle \frac{16}{8} \right\rangle = \langle 2 \rangle = \{0, 2, 4, 6, 8, 10, 12, 14\}$
$\left\langle \frac{16}{16} \right\rangle = \langle 1 \rangle = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\} = Z_{16}$.

Lemma (Intersection of Cyclic Subgroups Lemma)[5]: If a and b are elements of a group G, then the cyclic subgroups generated by a and b have a non-trivial intersection if and only if a and b have a common divisor greater than 1.

Example:

Consider Z12 = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}.

The factors of 12 are: k = {1, 2, 3, 4, 6, 12}.

The corresponding subgroups are:

$$\left\langle \frac{12}{1} \right\rangle = \langle 12 \rangle = \{0\}$$
$$\left\langle \frac{12}{2} \right\rangle = \langle 6 \rangle = \{0, 6\}$$
$$\left\langle \frac{12}{3} \right\rangle = \langle 4 \rangle = \{0, 4, 8\}$$
$$\left\langle \frac{12}{4} \right\rangle = \langle 3 \rangle = \{0, 3, 6, 9\}$$
$$\left\langle \frac{12}{6} \right\rangle = \langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$$
$$\left\langle \frac{12}{12} \right\rangle = \langle 1 \rangle = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} = Z_{12}.$$

Definition: The intersection graph of a group is a graph whose vertices are the non-trivial subgroups of G, and two distinct vertices H and K are connected if $H \cap K \neq \{e\}$, denoted by $\Gamma_{int}(G)$ [7].

Connectedness Methodology [1, 4]: Let Γ(V,E) be a graph with vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, e_3, \ldots, e_m\}$. We use the following algorithm to determine whether the graph Γ(V,E) is connected:

(1) Select any two distinct vertices vi, vj ∈ V , where i, j ∈ {1, 2, 3, . . . , 11} arbitrarily.
(2) Using the graph definition, check whether there exists a path connecting vi and vj.
(3) If a path exists, then the graph is connected; otherwise, it is not connected.

RESULT

Vertices Intersection	Connection
$\langle 4 \rangle \cap \langle 2 \rangle = \{0\}$	Not connected
$\langle 4 \rangle \cap \langle 1 \rangle = \{0\}$	Not connected
$\langle 2 \rangle \cap \langle 1 \rangle = \{0, 3\}$	Connected

Table 1 Vertices Connection Table

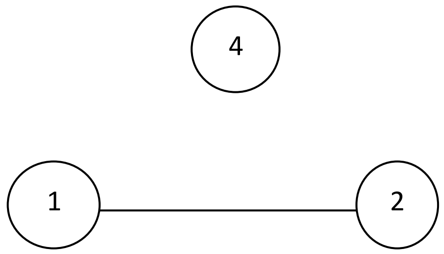

Figure 3.1: Intersection graph of Z_4

Vertices Intersection	Connection
⟨10⟩ ∩ ⟨5⟩ = {0}	Not connected
⟨10⟩ ∩ ⟨2⟩ = {0}	Not connected
⟨10⟩ ∩ ⟨1⟩ = {0}	Not connected
⟨5⟩ ∩ ⟨2⟩ = {0}	Not connected
⟨5⟩ ∩ ⟨1⟩ = {0, 5}	Connected
⟨2⟩ ∩ ⟨1⟩ = {0, 2, 4, 6, 8}	Connected

Table 2 Vertices Connection Table for Z_{10}

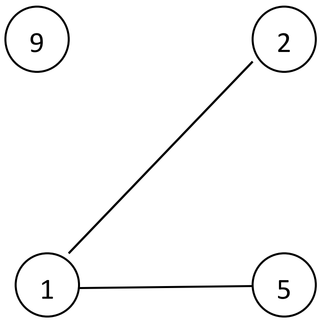

Figure 3.2: Intersection graph of Z_{10}

Vertices Intersection	Connection
⟨10⟩ ∩ ⟨5⟩ = {0}	Not connected
⟨10⟩ ∩ ⟨2⟩ = {0}	Not connected
⟨10⟩ ∩ ⟨1⟩ = {0}	Not connected
⟨5⟩ ∩ ⟨2⟩ = {0}	Not connected
⟨5⟩ ∩ ⟨1⟩ = {0, 5}	Connected
⟨2⟩ ∩ ⟨1⟩ = {0, 2, 4, 6, 8}	Connected

Table 3: Vertices Connection Table for Z_{10}

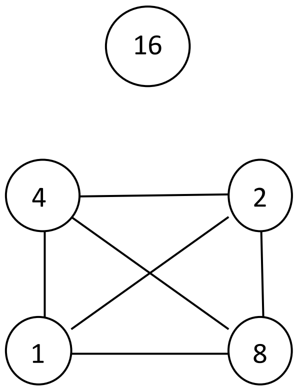

Figure 3.3: Intersection graph of Z_{16}.

We begin our result and discussion section by analyzing the structures of the intersection graphs of Zn in terms of finding the numbers of edges it contain for the different values of n considered. The following three theorems gives the structures of the sizes of these graphs.

DISCUSSION

Theorem 3.1 If n is a prime number, then $\Gamma_{int}(Zn)$ has no edge.

Proof: Assume n is prime, then n has only 2 positive divisors: 1 and n. By Theorem 1, Z_n has 2 unique subgroups which are:

$$\left\langle \frac{n}{1} \right\rangle = \langle n \rangle = \{0\} = \{0\}$$

$$\left\langle \frac{n}{1} \right\rangle = \langle 1 \rangle = \{Z_n\}$$

Since V $(\Gamma_{int}(Z_n))$ are the non-trivial subgroups of Zn, then $\Gamma_{int}(Z_n)$ contains only one vertex (Z_n) with no edge. Hence proved.

Theorem (3.1) reveals that when n is a prime number, the intersection graph $\Gamma_{int}(Z_n)$ has no edges. This result has profound implications for the structure of Z_n, underscoring its unique subgroup arrangement. The absence of edges in $\Gamma_{int}(Z_n)$ indicates that Z_n boasts a singular subgroup structure, devoid of intersections between its subgroups. This, in turn, reinforces the cyclic nature of Z_n, as the graph's emptiness reflects the group's simple, cyclic architecture. The derivation of Z_n's properties from the graph's properties is particularly insightful. The lack of edges in $\Gamma_{int}(Z_n)$ implies that Z_n is characterized by an absence of non-trivial intersections between its subgroups. Furthermore, the graph's emptiness suggests that Z_n's subgroup lattice is remarkably simple, with no complex relationships between subgroups. This simplicity is a direct consequence of the prime nature of n, which imposes a stringent structure on Z_n's subgroups. Theorem (3.1) is intimately aligned with the aim and objectives of this research. By investigating the graph-theoretic properties of Z_n, we gain a deeper understanding of the group's underlying structure. Theorem (3.1), in particular, provides a nuanced analysis of Z_n's subgroup arrangement, shedding light on the unique properties that emerge when n is prime. This result has significant implications for our comprehension of Z_n's behavior, particularly in the context of group theory and its applications.

Theorem 3.2 If n = pq, where p, q are distinct numbers, then $\Gamma_{int}(Z_n)$ contains only 2 edges.

Proof: Assume n = pq, then n has 4 factors: 1, p, q, and n, with n = pq and $p \neq q$. By Theorem 1, Z_n has the following subgroups:

$$\langle n \rangle = \{0\}$$
$$\left\langle \frac{n}{p} \right\rangle = \langle q \rangle$$
$$\left\langle \frac{n}{q} \right\rangle = \langle p \rangle$$

Z_n is adjacent to $\langle q \rangle$ because $Z_n \cap \langle q \rangle = \{q\} \neq \{0\}$.
Z_n is adjacent to $\langle p \rangle$ because $Z_n \cap \langle p \rangle = \{p\} \neq \{0\}$.
$\langle q \rangle$ is not adjacent to $\langle p \rangle$ because by Lemma 1, gcd(p, q) = 1, thus $\langle q \rangle \cap \langle p \rangle = \{0\}$.
Therefore, $\Gamma_{int}(Z_n)$ has only 2 edges: $Z_n \sim \langle p \rangle$ and $Z_n \sim \langle q \rangle$, Hence proved.

Theorem (3.2) presents a fascinating result, stating that if n = pq, where p and q are distinct prime numbers and n has only four factors (1, p, q, and pq), then the intersection graph $\Gamma_{int}(Z_n)$ contains exactly two edges. This theorem offers valuable insights into the structural properties of Z_n and its intersection graph. The presence of two edges in $\Gamma_{int}(Z_n)$ indicates that the subgroups of Z_n intersect in a highly structured manner. Specifically, the 2 edges represent the non-trivial intersections between the subgroups generated by p, q, and pq. This result highlights the intricate relationships between the subgroups of Z_n, which are dictated by the prime factorization of n. The derivation of Z_n's properties from the graph's properties is once again enlightening. The presence of 2 edges in $\Gamma_{int}(Z_n)$ implies that Z_n has a relatively simple subgroup lattice, with a limited number of intersections between subgroups. Furthermore, the graph's structure suggests that the subgroups of Z_n can be organized into a hierarchical arrangement, with the subgroups generated by p and q serving as the foundation. Theorem (3.2) is closely aligned with the aim and objectives of this research, as it provides a detailed analysis of the intersection graph of Z_n. By examining the structural properties of $\Gamma_{int}(Z_n)$, we gain a deeper understanding of the underlying relationships between the subgroups of Z_n. This result has significant implications for our comprehension of Z_n's behavior, particularly in the context of group theory and its applications.

Theorem 3.3 If n = p^2, then $\Gamma_{int}(Z_n)$ has only one edge.

Proof: Assume n = p^2, then n has only 3 factors: 1, p, and p^2.
By Theorem 1, Z_n has 3 unique subgroups which are:

$$\langle p^2 \rangle = \{0\}$$
$$\left\langle \frac{p^2}{p} \right\rangle = \langle q \rangle$$
$$\left\langle \frac{p^2}{p} \right\rangle = Z_n$$

$\langle p \rangle \sim Z_n$ because $p \in \langle p \rangle \cap Z_n$, with $p \neq 0$, and this is the only edge since there are only two non-trivial subgroups.
Hence, $\Gamma_{int}(Z_n)$ contains only 1 edge.

Theorem (3.3) presents an intriguing result, stating that if n = p^2, where p is a prime number, then the intersection graph $\Gamma_{int}(Z_n)$ has only one edge. This theorem provides a fascinating insight into the structural properties of Z_n and its intersection graph. The presence of only one edge in $\Gamma_{int}(Z_n)$ indicates that the subgroups of Z_n intersect in a highly restricted manner. Specifically, the single edge represents the non-trivial intersection between the subgroup generated by p and the subgroup generated by p^2. This result highlights the rigid structure of Z_n's subgroups, which is dictated by the prime factorization of n. The derivation of Z_n's properties from the graph's properties is once again enlightening. The presence of only one edge in $\Gamma_{int}(Z_n)$ implies that Z_n has a highly simplified subgroup lattice, with minimal intersections between subgroups. Furthermore, the graph's structure suggests that the subgroups of Z_n are organized in a hierarchical manner, with the subgroup generated by p serving as the foundation. Theorem (3.3) is closely aligned with the aim and objectives of this research, as it provides a detailed analysis of the intersection graph of Z_n. By examining the structural properties of $\Gamma_{int}(Z_n)$, we gain a deeper understanding of the underlying relationships between the subgroups of Zn. This result has significant implications for our comprehension of Z_n's behavior, particularly in the context of group theory and its applications. This result can be seen as a natural extension of Theorem (3.1), which states that if n is prime, then $\Gamma_{int}(Z_n)$ has no edges. In contrast to Theorem (3.2), which shows that n = pq results in a graph with four edges, Theorem (3.3) demonstrates that n = p^2 yields a graph with only one edge, highlighting the significant impact of the prime factorization of n on the structure of $\Gamma_{int}(Z_n)$. The next theorem discusses the element that is common to all non-trivial subgroups of Z_n for n = 2^k.

Theorem 3.4 If n = 2^k, then every non-trivial subgroup of Z_n contains the element 2^{k-1}.

Proof: Assume n = 2^k. Then, by Theorem 1, n has (k + 1) factors as follows:

$$1, 21, 22, 23, 24, \ldots, 2^k.$$

For each factor, we have a unique subgroup (by Theorem 1). Therefore, Z_n has (k+1) subgroups, one for each factor, and hence k non-trivial subgroups as follows:

$$\left\langle \frac{n}{2} \right\rangle = \left\langle \frac{2^k}{2} \right\rangle = \langle 2^{k-1} \rangle = \{0, 2^{k-1}\}$$
$$\left\langle \frac{n}{2^2} \right\rangle = \langle 2^{k-2} \rangle = \{0, 2^{k-2}, (2)2^{k-2}, (3)2^{k-2}\}$$

$$\left\langle \frac{n}{2^3} \right\rangle = \langle 2^{k-3} \rangle = \{0, 2^{k-3}, (2)2^{k-3}, \ldots, (7)2^{k-3}\}$$

:

:

$$\left\langle \frac{n}{2^k} \right\rangle = Z_n$$

This gives k non-trivial subgroups of Z_n. Claim: $2^{k-1} \in \left\langle \frac{n}{2^l} \right\rangle$, for all l = 1, 2, 3, . . . , k.
Consider a non-trivial subgroup of Z_n,

$$\left\langle \frac{n}{2^l} \right\rangle = \left\langle \frac{2^k}{2^l} \right\rangle = \langle 2^{k-l} \rangle,$$

for some l = 1, 2, 3, . . . , k.

Then $\left\langle \frac{n}{2^l} \right\rangle$ has 2^l elements for all l = 1, 2, 3, . . . , k:

$$\left\langle \frac{n}{2^l} \right\rangle = \langle 2^{k-l} \rangle = \{0, 2^{k-l}, (2)\, 2^{k-l}, (3)\, 2^{k-l}, \ldots, (2^l - 1)\, 2^{k-l}\}$$

Since $\langle 2^{k-l} \rangle$ consists of elements of the form (x) 2^{k-l} with x between 1 and 2^l, then:

$$\left(\frac{2l}{2}\right) 2^{k-l} \in \langle 2^{k-l} \rangle$$

Thus:

$$\left(\frac{2^l}{2}\right)(2^{k-l}) \in \langle 2^{k-l} \rangle$$
$$\Rightarrow (2^l - 1)(2^{k-l}) \in \langle 2^{k-l} \rangle.$$
$$\Rightarrow (2^l - 1 + k - l) = 2^{k-l} \in \langle 2^{k-l} \rangle.$$

Since $\langle 2^{k-l} \rangle$ is arbitrary, this proves our claim, Hence the result.

Theorem (3.4) presents an intriguing result, stating that if $n = 2^k$, then every non-trivial subgroup of Z_n contains the element 2^{k-l}. This theorem provides valuable insight into the structural properties of Z_n's subgroups. The presence of the element 2^{k-l} in every non-trivial subgroup of Z_n indicates that these elements play a crucial role in the subgroup structure of Z_n. Specifically, the elements 2^{k-l} serve as a kind of "skeleton" for the subgroups of Z_n, as every subgroup must contain these elements. The derivation of Z_n's properties from the subgroup structure is once again enlightening. The presence of the elements 2^{k-l}in every nontrivial subgroup of Z_n implies that Z_n has a highly structured subgroup lattice, with these elements serving as a foundation for the lattice. Theorem (3.4) is closely aligned with the aim and objectives of this research, as it provides a detailed analysis of the subgroup structure of Z_n. By examining the properties of Z_n's subgroups, we gain a deeper understanding of the underlying relationships between the elements of Z_n. This result has significant implications for our comprehension of Z_n's behavior, particularly in the context of group theory and its applications. This result highlights the importance of the prime factorization of n in determining the subgroup structure of Z_n. The fact that $n = 2^k$ leads to a highly structured subgroup lattice suggests that the prime factorization of n plays a crucial role in shaping the internal structure of Z_n.

CONCLUSION

In conclusion, this study has successfully investigated the structural properties of the intersection graph $\Gamma_{int}(Z_n)$ of the subgroups of Z_n, with a focus on the cases where n = p, n = pq, n = p2 (p prime), and $n = 2^k$ (k a natural number). The study aimed to determine the connectedness of the above graphs. The objectives of the study were fully achieved, as the research provided a comprehensive analysis of the structural properties of $\Gamma_{int}(Z_n)$ for the specified cases. The findings revealed unique and intriguing properties of the graph.

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