AJMS
Asian Journal of Mathematical Sciences

RESEARCH ARTICLE

BANACH'S FIXED POINT THEOREM ON THE EXISTENCE AND UNIQUENESS OF VOLTERRA INTEGRAL EQUATION SOLUTION

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Received: 09-04-2025; Revised: 15-05-2025; Accepted: 11-06-2025

ABSTRACT

An integral equation is an inversion of a differential equation. Therefore, an integral equation can be seen as the solution of a differential equation with a given condition at an interval. So that through this integral equation, it can be ensured that the existence of a solution to the differential equation can be ascertained. In this article, it is shown in a functional analysis how Banach's fixed-point theorem can show the existence and uniqueness of the solution of integral equations, so that it can guarantee the existence of the solution of a differential equation.

Keywords: Banach fixed point, Fredholm equation, and Volterra integral equation

INTRODUCTION

Banach's fixed-point theorem or often called Banach's theorem is a very important theorem in mathematical analysis, especially to show the existence and uniqueness the solution of differential equations [1]. The statement of this theorem is a certain form of mapping of a complete metric space to itself. As for the definition, fixed point in this case it is a mapping $T: X \rightarrow X$ in such a way that each element $x \in X$ mapped to itself in X that is

$TX = x$,

Map Tx coinciding with x [2].

Banach's theorem provides sufficient conditions for the existence and uniqueness of a fixed point for a class of mapping called contraction [3]. In this case it is defined as follows.

Definition Suppose $X = (X, d)$ is a metric space. A mapping of $T: X \rightarrow X$ is called a contraction on X if there is a real number$\alpha < 1$ such that for every $x, y \in X$ occurs

$d(Tx, Ty) \leq \alpha\, d(x, y)$, $\alpha < 1$

Banach's fixed-point theorem contains a contraction theorem in it. This theorem is a theorem of existence and uniqueness for fixed points of a given mapping, and provides a constructive iteration procedure to obtain the best approximation [4]. The statement of the theorem is stated in theorem 1.2 below [2],[3]

Theorem 1.2. Let the metric space X = (X, d) with X $\neq\emptyset$. Suppose X is complete and the mapping of T: X $\rightarrow$ X contractions on X, then T has exactly one fixed point.

Furthermore, consider the following integral equation:

$$x(t)-\mu\int_{a}^{b}k(t,\tau)x(\tau)d\tau=v(\tau)$$

the so-called equation **Fredholm** The second type[5]. In this case, the interval [a, b] given. x is the function on [a, b] unknown.μis a parameter. Kernel k is a continuous function given to the G = [a, b]×[a, b], andv is a function that is given and defined in [a, b].

Integral equations can be viewed in various function spaces. In this case it will be looked at C[a, b], i.e. the space of all continuous functions defined at the interval J = [a, b], with the metric d given by

$$d(x,y)=\max_{t\in J}|x(t)-y(t)|$$

In the case of the integral upper limit of equation (1.1) which is constant expressed as a variable $t\in[a, b]$, such that it becomes

$$x(t)-\mu\int_{a}^{t}k(t,\tau)x(\tau)d\tau=v(\tau),$$

So the equation (1.3) is called the integral equation **Volterra** [5]**.** As is known, the integral equation (1.3) is an inversion equation of a differential equation (2.5). Therefore the equation can be seen as a solution of a differential equation with the conditions given at an interval [a, b]. The problem is whether the integral equations, especially those stated in (1.3) above, have a solution? What is the role of Banach's theorem in guaranteeing the existence of the solution of the integral equation (1.3) mentioned above?

DISCUSSION:

To show how the role of Banach's theorem in guaranteeing the completion of the integral equation (1.3), consider the integral equation (1.1) above. In this case, suppose C [a, b] is complete. Assume $v\in C[a,b]$ that the kernel k is continuous on G, then k is a finite function on G, so that it can be stated that for all ,$(t,\tau)\in G$

$$|k(t,\tau)|\leq c$$

It is clear that equation (1.1) can be expressed by x = Tx, so it can be written

$$Tx(t)=v(\tau)+\mu\int_{a}^{b}k(t,\tau)x(\tau)d\tau \qquad (2.1)$$

Since v and k are continuous, then equation (2.1) defines an operator$T\colon C[a,b]\rightarrow C[a,b]$. Therefore, the parameter μ can be given a constraint in such a way that T becomes contractive. Based on (1.2) and (2.1), it can be stated that

$$d(Tx,Ty)=\max_{t\in J}|Tx(t)-Ty(t)|$$

$$=|\mu|\max_{t\in J}\left|\int_{a}^{b}k(t,\tau)[x(\tau)-y(\tau)]d\tau\right|$$

$$\leq |\mu| \max_{t \in J} \int_a^b |k(t,\tau)||x(\tau) - y(\tau)| d\tau$$

$$\leq |\mu| c \max_{t \in J} |x(\sigma) - y(\sigma)| \int_a^b d\tau$$

$$= |\mu| c d(x,y)(b-a)$$

So it is obtained$d(Tx - Ty) \leq \alpha d(x,y)$, with$\alpha = |\mu| c(b-a)$. It is shown T becomes contractive $(\alpha < 1)$provided that

$$|\mu| < \frac{1}{c(b-a)}$$

Based on the idea mentioned above, it can be shown that the integral equation (1.3) will have a unique solution x at [a,b] for every μ.

Suppose and $v \in C[a,b]$, and$k$ is continuous in the area of the plane of the triangle R in the field $t\tau$ given by$a \leq \tau \leq t, a \leq t \leq b$. If equation (1.3) is written with x = Tx for the mapping$T: C[a,b] \to C[a,b]$, i.e

$$Tx(t) = v(\tau) + \mu \int_a^t k(t,\tau) x(\tau) d\tau \ (2.2)$$

So by using equation (1.2), for all $x, y \in C[a,b]$ can be obtained

$$|Tx(t) - Ty(t)| = |\mu| \left| \int_a^t k(t,\tau)[x(\gamma) - y(\gamma) d\tau] \right|$$

$$\leq |\mu| c d(x,y) \int_a^t d\tau$$

$$= |\mu| c(t-a) d(x,y)$$

Because$T \in B(X,X)$, by induction it is obtained

$$|T^m x(t) - T^m y(t)| \leq |\mu|^m c^m \frac{(t-a)^m}{m!} d(x,y) \qquad (2.3)$$

For $t - a \leq b - a$ on the right segment of equation (2.3) and then by taking the upper minimum $t \in J$ at the left end of the interval, then from equation (2.3) is obtained

$$d(T^m x, T^m y) \leq \alpha_m d(x, y)$$

Where

$$\alpha_m = |\mu|^m c^m \frac{(b-a)^m}{m!}$$

In this case for every μ and m that is large enough, will be obtained . As a result, it contracted on $\alpha_m < 1$. In the case of $T^m C[a, b]$ contraction $m \in Z^+$, then the mapping $T: X \to X$ in the completee metric space $X = (X, d)$ has a unique fixed point. Proof of this statement can be shown as follows,

Assume $B = T^m$ the contradiction in X. According to theorem 1.2 above, mapping B has a unique fixed point $\hat{x}$, i.e. $B\hat{x} = \hat{x}$. Therefore $B^n\hat{x} = \hat{x}$. This is as a result of Banach's fixedpoint theorem that for every $x \in X$ and $n \to \infty$ is obtained $B^n\hat{x} \to \hat{x}$. Especially if $x = T\hat{x}$, and $B^n = T^{nm}$, then obtained

$$\begin{aligned} \hat{x} = \lim_{n\to\infty} B^n T\hat{x} &= \lim_{n\to\infty} TB^n\hat{x} \\ &= \lim_{n\to\infty} T\hat{x} \\ &= T\hat{x} \end{aligned}$$

This shows that $\hat{x}$ it is a fixed point of T . Since every fixed point of T is also a fixed point B, it appears that T cannot have more than one fixed point. Therefore, based on the above description, the existence of a solution of equation (1.3) is shown.

The uniqueness of the solution of equation (1.3) above can be shown through the following explanation,

Suppose X is a Banach space, and assume that K is the operator on X(i.e. the mapping of X on itself)[6], such that K is the sequencer operator for each , then therex is a unique on X such that for each parameter μ applies

$$x = v + \mu K x \tag{2.4}$$

This equation (2.4) is a form of simplification of equation (1.3) by applying the operator K as the integral operator in the second quarter of equation (1.3) which is acted on x[7].

For example, there are two solutions of the equation (2.4), x_1 and x_2. Suppose $x = x_1 - x_2$, then there is a $\{x_n\}$ sequence on X in such a way that

$$x_n = v + \mu K x_{n-1}$$

That is

$$x_1 = v + \mu K x_0$$

$$\begin{aligned} x_2 &= v + \mu K x_1 \\ &= v + \mu K(v + K x_0) \end{aligned}$$

$$= v + \mu K v + \mu K^2 x_0$$

in this case$K^2x = K(Kx)$. If it is defined that$K^n x = K(K^{n-1}x)$, then it is inductively seen that the action of the$K$ operator acted on $x \in X$will producethe result of a map of$x \in X$, consequently as stated in Banach's theorem it can be stated that$x = \mu Kx$. So that for every$n$ will be obtained

$$x = \mu K^2 x = \mu K^3 = \cdots = \mu K^n x$$

Because $K \in B(X, X)$, then for $n \to \infty$

$$\|x\| = \|\mu K^n x\| = |\mu| \|K^n x\| \to 0$$

In this case x is independent of n. So if and only if $\|x\| = 0$x = 0. This means that . It is shown that the settlement of (2.4) is singular.$x_1 = x_2$.

From the above description, it can be seen that the Banach fixed point is a sufficient condition for existence and uniqueness for integral equations. In this case, an integral equation is an inversion equation of an explicit ordinary differential equation, especially the first order which is sequencer. With the condition of the fixed point, it can be seen that the solution of the integral equation forms an iterative sequence.

Consider the first order explicit differential equation with the following initial conditions [8],

$x' = f(t, x)$ with initial conditions$x(t_0) = x_0$ (2.5)

In this case f is continuous on$R = \{(t, x)P: |t - t_0| \le a, |x - x_0| \le b\}fora, b \in P$, then f is limited to R, so$|f(t, x)| \le c$. If satisfies the Lipchitz condition in R, i.e. for every $(t; x), (t; v) \in R$then there is such a positive real k constant that it applies

$|f(t, x) - (t, v)| \le k(x - v)$(2.6)

then the initial value problem of equation (2.5) has a unique solution at intervals$[t_0 - \beta, t_0 + \beta]$ with$\beta < min\left\{a, \frac{b}{c}, \frac{1}{k}\right\}$. This can be proven as follows:

Proof. For example, C(J) is the metric space of all real-value continuous functions at intervals $J = [t_0 - \beta, t_0 + \beta]$with metric defined by (1.2), so that C(J) is complete. Suppose, $\tilde{C}$ is a subspace of C (J) contains all the functions x $\in$C (J) that satisfies$|x(t) - x_0| \le c\beta$, so$\tilde{C}$ is a complete space. By integrating equation (2.5) and using Banach's fixed-point theorem x = Tx with$T: \tilde{C} \to \tilde{C}$, i.e.

$$Tx(t) = v(\tau) + \int_{t_0}^{t} f(\tau, x(\tau)) d\tau \qquad (2.7)$$

T is defined for every $x \in \tilde{C}$. Because$c\beta < bc$, so if$x \in \tilde{C}$ then$\tau \in J$ and$(\tau, x(\tau)) \in R$, furthermore integral (2.7) exists fis continuous in R. Hence, it is obtained

$$|Tx(t) - x_0| = \left| \int_{t_0}^{t} f(\tau, x(\tau)) d\tau \right| \le c|t - t_0| \le c\beta$$

Further for T on$\tilde{C}$ with the condition of Lipschitz obtained

$$|Tx(t) - Tv(t)| = \left| \int_{t_0}^{t} [f(\tau, x(\tau)) - f(\tau, v(\tau))] d\tau \right|$$

$$\leq |t-t_0|\max_{t\in J} k|x(\tau)-v(\tau)|$$

$$\leq k\beta d(x,v)$$

Since the right segment does not depend on t, the maximum value is obtained from the left segment, which is $|Tx-Tv| \leq \alpha d(x,v)$ by $\alpha = k\beta$. In order to T contraction $\tilde{C}$, it must be $\beta < min\left\{a,\frac{b}{c},\frac{1}{k}\right\}$. Therefore, according to Banach's fixed point theorem guarantees T has a unique fixed point $x\in\tilde{C}$. Proof is complete.

From the above proofing it can be seen that a solution of equation (2.7) is a continuous function in x at J that satisfies x = TX in such a way that the solution of equation (2.7) is guaranteed to exist and is singular. Hence the solution x of equation (2.7) forms a convergent sequence of the iteration, $\{x_n\}$

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(\tau, x_n(\tau))d\tau \tag{2.8}$$

For n = 0, 1, 2, ...

For example, the differential equation $y' = y - 1$ with the initial value $y(0) = 2$. The solution of the differential equation is an integral equation (2.7) that forms a sequence of iterative (2.8) in such a way that with $x_0 = 0$ and $y(x_0) = 2$ obtained

$$y_{n+1}(t) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt$$

By $f(t, y_n(t)) = y_n(t) - 1$. So that the results of the iteration calculation are obtained as follows

$$y_1(t) = 2 + \int_0^x (2-1)dt$$

$$= 2 + \int_0^x dt$$

$$= 2 + x$$

$$y_2(t) = 2 + \int_0^x (2+x-1)dt$$

$$= 2 + \int_0^x (1+x)dt$$

$$= 2 + x + \frac{1}{2}x^2$$

$$y_3(t) = 2 + \int_0^x \left(2 + x + \frac{1}{2}x^2 - 1\right)dt$$

$$= 2 + \int_0^x \left(1 + x + \frac{1}{2}x^2\right) dt$$

$$= 2 + x + \frac{1}{2}x^2 + \frac{1}{2.3}x^3$$

$$y_4(t) = 2 + \int_0^x \left(2 + x + \frac{1}{2}x^2 + \frac{1}{2.3}x^3 - 1\right) dt$$

$$= 2 + \int_0^x \left(1 + x + \frac{1}{2}x^2 + \frac{1}{2.3}x^3\right) dt$$

$$= 2 + x + \frac{1}{2}x^2 + \frac{1}{2.3}x^3 + \frac{1}{2.3.4}x^4$$

It appears that the above iteration sequence$\{y_n\}$ will converge$1 + e^x$, so it can be concluded that the solution of$y' = y - 1$ with the initial value$y(0) = 2$ is$1 + e^x$.

CONCLUSION

The fixed point of the operator can be determined by forming a sequence of contractive iterations. A contractive sequence can be obtained if the operator is contractive, and Banach's fixed-point theorem guarantees that the point is excist and unique. The existence and uniqueness occur in the solution of the Volterra integral equation which can be applied as a solution of the explicit first order linear differential equation with the initial conditions.

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