AJMS
Asian Journal of Mathematical Sciences

REVIEW ARTICLE

Numerical Range Distortion Under Unitary Equivalence

Faith Mwangi Muthoni[1*] **Benard M. Nzimbi[2]** **Stephen W. Luketero[3]**

Department of Mathematics, University of Nairobi

Corresponding Email: mwangifm@uonbi.ac.ke

Received: 03-04-2025; Revised: 11-05-2025; Accepted: 05-06-2025

ABSTRACT

This paper investigates the preservation properties of numerical ranges under unitary equivalence transformations in operator theory. We establish that if operators T and S are unitarily equivalent via $S = U^* TU$, then their numerical ranges are identical: $W(S) = W(T)$. Beyond this fundamental equality, we prove that unitary equivalence preserves critical geometric properties of numerical ranges, including extreme points, exposed points, supporting lines, contact points, and the geometric multiplicity of boundary points. For normal operators, we demonstrate that numerical range equality characterizes unitary equivalence, providing a geometric criterion for this algebraic relation. Additionally, we show that the curvature of the boundary of numerical ranges remains invariant under unitary transformations. These results highlight the deep connection between the algebraic structure of operators and the geometric properties of their numerical ranges, contributing to our understanding of operator behavior under unitary equivalence.

Keywords: Numerical Range, Unitary Equivalence, Normal Operators

INTRODUCTION

The study of numerical ranges and operator behavior under various equivalence relations represents a fundamental area in operator theory[1]. Building on the seminal work of Toeplitz (1918), the numerical range has proven to be a powerful tool for understanding operator behavior. Following Halmos (1962) and Goldberg and Tadmor (1982), we begin with the foundational definition:

Definition 1.1: (Numerical Range) For $T \in B(H)$, the numerical range is defined as:

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on H[2,3].

The geometric properties of numerical ranges provide crucial insights into operator behavior[4]. These properties often reveal deeper structural aspects of operators that may not be immediately apparent from algebraic considerations[5]. Our study focuses on how these properties transform under unitary equivalence.

Definition 1.2: (Unitary Equivalence). Operators T, S are unitarily equivalent if there exists a unitary operator U such that $S = U^{*}TU$

Definition 1.3: (Supporting Lines and Contact Points). A supporting line of a convex set C is a line that intersects C but does not pass through its interior[6]. The points where the supporting line touches C are called contact points. Supporting lines provide a geometric way to describe the boundary structure of W (T), as they reveal how the numerical range interacts with its enclosing convex hull. This concept is fundamental to understanding the boundary behavior of numerical ranges, as emphasized by Stampfli (1970).

Example 1.4: Suppose $W(T)$ is an elliptical region in the complex plane. Each supporting line corresponds to a tangent to the ellipse, and the contact points are the points of tangency. These points are crucial for understanding the geometry of $W(T)$, as they determine the shape and orientation of the ellipse. For instance, if:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

then the supporting lines of W (T) = [1, 2] are vertical lines at 1 and 2, with contact points corresponding to the eigenvalues 1 and 2. This observation is supported by the work of Ando (1973)[7].

Definition 1.5: (Extreme Points). An extreme point z of a convex set C is a point that cannot be expressed as a convex combination of other points in C. That is,

$$z \neq tx + (1-t)y$$

for $x, y \in C, t \in (0, 1)$

In the context of the numerical range W (T), extreme points often correspond to eigenvalues. When operator T has no eigenvalues, extreme points of the numerical range W (T) correspond to:

1. **Approximate eigenvalues:** $\lambda \in \sigma_{\text{ap}}(T)$ where $\exists\{x_n\}$ with $\|x_n\| = 1$ and $\|(T - \lambda I)x_n\| \to 0$
2. **Boundary spectrum points**: By Donoghue Jr (1957) theorem, extreme points lie in σ(T)
3. **Supporting hyperplane zeros**: Point z is extreme iff $e^{i\phi}(T - zI) + e^{-i\phi}(T - zI)^{*}$ has zero eigenvalue for some ϕ
4. **Geometric configurations:** For shift operators, extreme points correspond to boundary of spectral disk

The geometric definition remains unchanged; spectral interpretation shifts from exact to approximate eigenstructure.
For normal operators, the extreme points of W (T) coincide with the eigenvalues of T lying on the boundary of the convex hull of the spectrum, as noted by Halmos (1962).

Example 1.6. Consider the diagonal matrix:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

The numerical range W (T) is the interval [1, 2], and the extreme points are 1 and 2, which are precisely the eigenvalues of T . This result aligns with the spectral properties discussed in Horn and Johnson (1991).

Definition 1.7: (Exposed Points). An exposed point of a convex set C is a point that lies uniquely on a supporting line of C. Exposed points are therefore "visible" from outside the set, as they are the unique points of contact between C and some supporting line. In the context of the numerical range W (T), exposed points are boundary points that are uniquely determined by the geometry of W (T), as explored in Gustafson (1974).

Example 1.8. Consider the operator:

$$T = \begin{matrix} 1 & 0 \\ 0 & i \end{matrix}$$

The numerical range W (T) is the line segment 1, i , in the complex plane. The endpoints 1 and i are both extreme points and exposed points because they lie on unique supporting lines.

Definition 1.9: (Geometric Multiplicity of Boundary Points). The geometric multiplicity of a boundary point $\lambda \in \partial W(T)$ refers to the dimension of the set of unit vectors x satisfying:

$$\langle Tx, x\rangle = \lambda.$$

This set is called the geometric fiber of λ, denoted by:

$M\lambda(T) = \{x : \|x\| = 1, \langle Tx, x\rangle = \lambda\}$.

The geometric multiplicity reflects how "rich" the boundary point is in terms of the underlying vector space structure. This concept has been studied extensively in the context of operator theory, particularly in Kendall (1975).

Example 1.10. Consider the operator:

$$T = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} .$$

Here, W {T} = {1}, and every unit vector x satisfies Tx, x = 1. Thus, M1(T) spans the entire unit sphere, and the geometric multiplicity is infinite. This example highlights the connection between geometric multiplicity and the degeneracy of eigenvalues, as discussed in Bhatia (1997).

Main Results

Theorem 2.1 (Equality of Numerical Ranges Under Unitary Equivalence). If T and S are unitarily equivalent via $S = U^*TU$, then their numerical ranges are equal:

W (T) = W (S).

Proof. Let $x \in H$ be a unit vector (x = 1). For $S = U^*TU$, compute:

$$\langle Sx, x\rangle = \langle U \; TUx, x\rangle = \langle TUx, Ux\rangle .$$

Since U is unitary, it preserves the norm, so Ux = x = 1. Moreover, the map x Ux is a bijection on the unit sphere. Thus:

$$\{\langle Sx, x\rangle : \|x\| = 1\} = \{\langle Ty, y\rangle : \|y\| = 1\}.$$

This shows that W (S) = W (T), completing the proof.

Example 2.2: Consider the matrices:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad S = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

These operators are unitarily equivalent via the permutation matrix:

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Both T and S have the same numerical range:

$$W(T) = W(S) = [1, 2].$$

This example demonstrates that even when the eigenvalues are permuted, the numerical range remains unchanged under unitary equivalence.

Theorem 2.3: (Preservation of Extreme and Exposed Points). Under unitary equivalence S = U ∗ TU, the extreme points and exposed points of the numerical range W (T) are preserved. That is, any extreme point or exposed point of W (T) corresponds to an extreme point or exposed point of W (S), respectively.

Proof. Let λ be an extreme point of W (T). Then there exists a unit vector y such that:

$$\langle Ty, y \rangle = \lambda,$$

and λ cannot be expressed as a convex combination of other points in W (T). For x = U ∗y, we have:

$$\langle Sx, x \rangle = \langle TUx, Ux \rangle = \langle Ty, y \rangle = \lambda.$$

If λ were not an extreme point of W (S), it would contradict the extremality of λ in W (T). A similar argument applies to exposed points, as they correspond uniquely to supporting lines, which are also preserved under unitary equivalence.

Example 2.4: Consider the operator:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

The numerical range W (T) is the convex hull of 1, i} , forming a line segment in the complex plane. The endpoints 1 and i are both extreme and exposed points. Under unitary equivalence via:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -I & i \end{pmatrix}$$

the transformed operator S = U ∗ TU has the same numerical range W (S) = W (T), and the endpoints 1 and i remain extreme and exposed points.

Theorem.2.5 (Preservation of Supporting Lines and Contact Points). Under unitary equiv- alence S = U ∗ TU, the supporting lines of the numerical range W (T) and their contact points are preserved. Specifically, any supporting line of W (T) corresponds to a supporting line of W (S), and the contact contact points remain identical.

Proof. A supporting line at angle θ corresponds to the minimum eigenvalue of the Hermitian operator:

$e^{-i\vartheta}T + e^{i\vartheta}T^*$.

For S = U ∗TU , this operator becomes:

$$e^{-i\vartheta}S + e^{i\vartheta}S^* = U^*(e^{-i\vartheta}T + e^{i\vartheta}T^*)U.$$

Since unitary transformations preserve eigenvalues and eigenvectors, the minimum eigenvalue and its corresponding eigenvector remain unchanged. Thus, the supporting lines and their contact points are preserved.

Example 2.6: Consider the operator:

$$T = \begin{matrix} 1 & 0 \\ 0 & 2 \end{matrix}$$

The numerical range W (T) = [1, 2] has vertical supporting lines at 1 and 2. Under unitary equivalence via:

$$U = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

the transformed operator S = U ∗TU has the same numerical range W (S) = [1, 2], and the supporting lines at 1 and 2 remain identical

Theorem 2.7: (Preservation of Geometric Multiplicity of Boundary Points). Under unitary equivalence S = U ∗ TU, the geometric multiplicity of boundary points of the numerical range is preserved. For any boundary point λ ∈ ∂W (T), the dimension of the set of unit vectors x satisfying ⟨ Tx, x⟩ = λ equals the corresponding dimension for S.

Proof. The geometric fiber of a boundary point λ is defined as:

$$M_\lambda(T) = \{x : \|x\| = 1, \langle Tx, x\rangle = \lambda\}.$$

Under unitary equivalence S = U ∗TU , the transformation x '→ Ux maps Mλ(T) bijectively onto Mλ(S). Since U is unitary, it preserves norms and inner products, ensuring that the dimension of Mλ(T) equals that of Mλ(S).

Example 2.8: Consider the operator:

$$T = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}.$$

The numerical range W (T) = {1} consists of a single point, and every unit vector x satisfies Tx, x = 1. Under unitary equivalence via any unitary matrix U , the transformed operator S = U ∗TU has the same numerical range W (S) = {1} , and the geometric multiplicity remains infinite.

Theorem 2.9: (Spectral Consequences of Numerical Range Equality for Normal Operators).

For normal operators, unitary equivalence is characterized by numerical range equality:

$$T \textit{ normal}, \; W(T) = W(S) \implies T \cong S,$$

where ∼= denotes unitary equivalence.

Proof. We proceed in several steps, utilizing the special properties of normal operators.

Since T is normal ($TT^* = T^*T$), the spectral theorem guarantees that there exists a spectral measure E_T such that:

$$T = \int_{\sigma(T)} \lambda \, dE_T(\lambda),$$

where $\sigma(T)$ denotes the spectrum of T. Similarly, for S, we have:

$$S = \int_{\sigma(S)} \mu \, dE_S(\mu).$$

For a normal operator T , the numerical range W (T) coincides with the convex hull of its spectrum:

$$W(T) = \mathrm{conv}(\sigma(T)).$$

This result follows from the spectral theorem and the fact that:

$$\langle Tx, x\rangle = \int_{\sigma(T)} \lambda \, d\langle E_T(\lambda)x, x\rangle,$$

for any unit vector x. A similar statement holds for S.

Given W (T) = W (S), it follows that:

conv(σ(T)) = conv(σ(S)).

Thus, the spectra σ(T) and σ(S) must coincide as sets, including multiplicities.

Let {λk}n denote the distinct eigenvalues of T (finite or countably infinite). For each k,

define the eigenspaces:

$$E_k = \ker(T - \lambda_k I), \quad F_k = \ker(S - \lambda_k I).$$

Since W (T) = W (S), the geometric multiplicity of each eigenvalue is preserved. Therefore:

dim(Ek) = dim(Fk),

for all k.

Define a unitary operator U : H → H by:

$$U = \sum_k U_k,$$

where each Uk : Ek → Fk is a unitary operator mapping the eigenspace Ek of T onto the eigenspace Fk of S.

For any x ∈ Ek, we have:

$$(U^*SU)x = U^*S(U_kx) = U^*(\lambda_kU_kx) = \lambda_kx = Tx.$$

Since {Ek} spans H (by the spectral decomposition of T), it follows that:

$U^*SU = T.$

The construction of U ensures that it is unique up to unitary operators commuting with T , as follows from the spectral theorem. Thus, T and S are unitarily equivalent.

Therefore, if T is normal and W (T) = W (S), then T ∼= S

Example 2.10. Consider the following non-diagonal normal matrices**:**

$$T = \begin{pmatrix} 1 & i \\ -I & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & -i \\ I & 1 \end{pmatrix}$$

Both T and S are normal because they satisfy TT ∗ = T ∗T and SS∗ = S∗S. Their eigenvalues can be computed as follows:

$\sigma(T) = \sigma(S) = \{1 + i, 1 - i\}.$

The numerical ranges of T and S are identical and are given by the convex hull of their eigenvalues:

$W(T) = W(S) = \{z \in \mathbb{C} : \mathrm{Re}(z) = 1, \mathrm{Im}(z) \in [-1, 1]\}.$

To explicitly construct the unitary equivalence, note that the eigenvectors of T and S correspond to their eigenvalues. For T , the eigenvectors associated with λ = 1 + i and λ = 1 i are:

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

For S, the eigenvectors are:

$$w_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad w_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Define the unitary operator U that maps the eigenvectors of T to those of S:

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It can be verified that S = U ∗TU , demonstrating that T and S are unitarily equivalent. This example illustrates the theorem’s conclusion for non-diagonal normal matrices.

Definition 2.11 (Curvature of the Boundary of a Numerical Range). The curvature κT (λ) of the boundary ∂W (T) at a point λ measures how rapidly the tangent direction changes as one moves along the boundary near λ.

For smooth portions of ∂W (T), the curvature can be computed using differential geometry techniques applied to the parametric representation of ∂W (T). If z(t) parameterizes ∂W (T) in the complex plane, the curvature is given by:

$$\kappa_T(\lambda) = \frac{|z'(t)|}{|z''(t)|},$$

evaluated at the point $\lambda = z(t0)$.

Proposition 2.12: (Preservation of Boundary Curvature Under Unitary Equivalence). Un- der unitary equivalence $S = U * TU$, the curvature of the boundary $\partial W(T)$ at any point is preserved. Specifically, if $\kappa T(\lambda)$ denotes the curvature of $\partial W(T)$ at a boundary point λ, then**:**

$$\kappa_T(\lambda) = \kappa_S(\lambda),$$

where $\kappa S(\lambda)$ is the curvature of $\partial W(S)$ at the corresponding boundary point.

Proof. To prove the proposition, we proceed as follows:

$$z_T(t) = \langle Tx(t), x(t)\rangle,$$

where x(t) is a parameterization of the unit sphere $\|x\| = 1$.

The curvature $\kappa T(\lambda)$ at a boundary point $\lambda = zT(t0)$ is given by:

$$\kappa_T(\lambda) = \frac{|z_T'(t)|}{|z_T''(t)|},$$

evaluated at $t = t0$.

For $S = U * TU$, compute $\partial W(S)$ using the same parameterization x(t). Since $S = U * TU$, we have:

$$z_S(t) = \langle Sx(t), x(t)\rangle = \langle TUx(t), Ux(t)\rangle = z_T(t).$$

Thus, $\partial W(S)$ coincides with $\partial W(T)$, and their parametric representations are identical.

Because $zS(t) = zT(t)$, the first and second derivatives of $zS(t)$ match those of $zT(t)$. There- fore:

$$\kappa_S(\lambda) = \frac{|z_S'(t)|}{|z_S''(t)|} = \frac{|z_T'(t)|}{|z_T''(t)|} = \kappa_T(\lambda).$$

This shows that the curvature at any boundary point is preserved under unitary equivalence. Thus, the conjecture is proven.

Example 2.13: Consider the normal matrix**:**

$$T = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

The numerical range $W(T)$ is the vertical line segment:

$$W(T) = \{z \in \mathbb{C} : \mathrm{Re}(z) = 1, \mathrm{Im}(z) \in [-1, 1]\}.$$

The boundary $\partial W(T)$ consists of two endpoints ($1+i$ and $1-i$) and a straight line connecting them. The curvature at all interior points of $\partial W(T)$ is zero, reflecting the flatness of the line segment.

Under unitary equivalence via:

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the transformed operator $S = U * TU$ has the same numerical range $W(S) = W(T)$, and the curvature remains zero along the entire boundary.

CONCLUSION

In this study, we have shown that unitary equivalence preserves not only the numerical range of an operator but also its deeper geometric and spectral characteristics. The invariance of extreme points, exposed points, supporting lines, and geometric multiplicity highlights the robustness of the numerical range as a unitary invariant. Notably, for normal operators, equality of numerical ranges fully characterizes unitary equivalence, underscoring the strong link between spectral properties and geometric representations. These results reinforce the significance of numerical ranges in operator theory, particularly as a tool for understanding equivalence classes and invariant features of bounded linear operators on Hilbert spaces.

REFERENCES

1. Ando, T. (1973). Structure of operators with numerical radius one. Acta Scientiarum Math- ematicarum (Szeged), 34, 11–15.

2. Bhatia, R. (1997). Matrix analysis. Springer.

3. Donoghue Jr, W. F. (1957). On the numerical range of a bounded operator. Michigan Math- ematical Journal, 4 (3), 261–263

4. Goldberg, M., & Tadmor, E. (1982). On the numerical radius and its applications. Linear Algebra and its Applications, 42, 263–284.

5. Gustafson, K. E. (1974). The numerical range and unitary equivalence. Linear Algebra and its Applications, 8, 25–35.

6. Halmos, P. R. (1962). A Hilbert space problem book. Van Nostrand.

7. Horn, R. A., & Johnson, C. R. (1991). Topics in matrix analysis. Cambridge University Press.

8. Kendall, D. G. (1975). The geometry of numerical ranges. Journal of Functional Analysis,

20, 1–12.

9. Stampfli, J. G. (1970). Hyponormal operators and spectral density. Transactions of the Amer- ican Mathematical Society, 149, 461–476.

10. Toeplitz, O. (1918). Das algebraische Analogon zu einem Satze von Fej'er. Mathematische Zeitschrift, 2 (1-2), 187–197

Let T denote the class of function $f(z)$ defined by

$$f(z)=z-\sum_{k=2}^{\infty}a_k z^k, \qquad (a_k \geq 0, k \in N=\{1,2,\cdots\}) \tag{1.1}$$

Which are univalent and analytic in the open unite disc $U=\{z:|z|<1\}$.

Definition 1.1: The class of starlike function of order μ denote by $S^*(\mu)$ if $f(z)\in T$ and satisfies the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\mu, \quad (0\leq\mu<1; z\in U) \tag{1.2}$$

Definition 1.2 : The class of convex function of order μ denote by $C(\mu)$ if $f(z)\in T$ and satisfies the condition

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\mu, \qquad (0\leq\mu<1, z\in U). \tag{1.3}$$

Note that $S^*(0)\equiv S^*$ is the class of starlike functions and $C(0)\equiv C$ is the class of convex functions.

Ruscheweyh [5] defined the differential operator $R^n f(z)$ as follows

$$R^0 f(z)=f(z),$$
$$R^1 f(z)=zf'(z)$$
$$(\delta+1)R^{\delta+1}f(z)=z\left(R^{\delta}f(z)\right)'+nR^{\delta}f(z)=z+\sum_{k=2}^{\infty}\delta(n,K)a_k z^k, \tag{1.4}$$

where $n\in N_0=N\bigcup\{0\}, z\in U$ and

$$\delta(n,K)=\frac{(n+k-1)!}{n!(k-1)!}. \tag{1.5}$$

Al-Oboudi [2] defined the differential operator $D_\lambda^n f(z)$ by:

$$D_\lambda^0 f(z)=f(z),$$
$$D_\lambda f(z)=D_\lambda f(z)=(1-\lambda)f(z)+\lambda zf'(z)$$
$$D_\lambda^n f(z)=D_\lambda\left(D_\lambda^{n-1}f(z)\right)=z-\sum_{k=2}^{\infty}[1+(k-1)\lambda]^n a_k z^k, \quad (n\in N_0, z\in U), \tag{1.6}$$

Lupas [3] defined the generalized differential operator $RD_{\lambda,\alpha}^n f(z)$ as linear combination of Ruscheweyh operator and Al-Oboudi differential operator by:

$$RD_{\lambda,\alpha}^n f(z)=(1-\alpha)R^n f(z)+\alpha D_\lambda^n f(z). \tag{1.7}$$

By simple calculate, we have

$$RD_{\lambda,\alpha}^{n} f(z)=z-\sum_{k=2}^{\infty}\left[\alpha[1+(k-1)\lambda]^{n}+(1-\alpha)\delta(n,k)\right]a_{k} z^{k} \quad (1.8)$$

From equation (1.7), we note that

$$RD_{\lambda,n}^{1} f(z)=z f'(z) \quad (1.9)$$

Now, by taking different value of the parameters n,λ,α and μ, we get some special cases of the operator $RD_{\lambda,\alpha}^{n} f(z)$, for example.

i. $RD_{\lambda,1}^{n} f(z)=D_{\lambda}^{n} f(z)$ studied by Al-Oboudi [2];
ii. $RD_{1,1}^{n}=S^{n} f(z)$, studied by Sâlâgean [6];
iii. $RD_{\lambda,0}^{n} f(z)=R^{n} f(z)$, studied by Ruscheweyh [5].

In 2014, Lupas and Andrei [4] use the generalized differential operator $RD_{\lambda,\alpha}^{n} f(z)$ to define the class $S_{\lambda,\alpha}^{n}(\mu)$, which consists of all function $f(z)\in T$ satisfies the condition

$$\operatorname{Re}\left\{\frac{z\left(D_{\lambda,\alpha}^{n} f(z)\right)'}{D_{\lambda,\alpha}^{n} f(z)}\right\}>\mu, \quad (z\in U; 0\le\mu<1), \quad (1.10)$$

where $RD_{\lambda,\alpha}^{n} f(z)$ given by (1.8) .

By specializing the parameters n,λ,α and μ, in the definition of the class $S_{\lambda,\alpha}^{n}(\mu)$ can be reduced know classes :

i. $S_{\lambda,\alpha}^{0}(\mu)=T^{*}(\mu)$ studied by Silverman [7];
ii. $S_{\lambda,\alpha}^{1}(\mu)=C(\mu)$ studied by Silverman [7];
iii. $S_{\lambda,0}^{n}(\mu)=S_{\lambda}^{*}(\mu)$ studied by Ahuja [1];
iv. Put $\alpha=0$, we get the class defined as follows:
v. $$\operatorname{Re}\left\{\frac{z\left(D_{\lambda}^{n} f(z)\right)'}{D_{\lambda}^{n} f(z)}\right\}>\mu, \quad (n\in N_{0}, z\in U; 0\le\mu<1);$$
vi. Put $\alpha=1$ and $\lambda=1$, we get the class defined as follows
vii. $$\operatorname{Re}\left\{\frac{z\left(S^{n} f(z)\right)'}{S^{n} f(z)}\right\}>\mu, \quad (z\in U; 0\le\mu<1).$$

The problem of coefficient estimates is one of interesting problems which was studied by researchers for certain classes in the open unit disc. Closely related to this problem Using the results of Lupas and Andrei [4] to determine radius of star likeness and radius of convexity details with some application of computers software .

RADII OF STARLIKENESS AND CONVEXITY

In order to prove our results, we need the following Lemma due to Lupas and Andrei [4] :

Lemma 2.1:

Let the function $f(z)$ defined by (1.1) belong to the class T, if

$$\sum_{k=2}^{\infty}(k-\mu)\left[\alpha[1+(k-1)\lambda]^n+(1-\alpha)\delta(n,k)\right]a_k \le 1-\mu, \tag{2.1}$$

then $f(z)\in S^n_{\lambda,\alpha}(\mu)$, where $0\le\mu<1$ and $\delta(n,k)$ defined by (1.5). The result is sharp for the function

$$f(z)=z-\frac{1-\mu}{(k-\mu)\left[\alpha[1+(k-1)\lambda]^n+(1-\alpha)\delta(n,k)\right]}z^k, \quad (k\ge 2). \tag{2. 2}$$

Now we study radius of starlikeness for the function $f(z)\in T$ belong to the classes $S^n_{\lambda,\alpha}(\mu)$ by obtaining the coefficient estimates.

Theorem 2.1:

Let the function $f(z)$ given by (1.1) be in the class $S^n_{\lambda,\alpha}(\mu)$,then $f(z)$ is starlike of order $\rho(0\le\rho<1)$ in $|z|<r_1(n,k,\lambda,\alpha,\mu,\rho)$, where

$$r_1(n,k,\lambda,\alpha,\mu,\rho)=\inf_{k\ge 2}\left[\frac{(1-\rho)(k-\mu)\left[\alpha[1+(k-1)\lambda]^n+(1-\alpha)\delta(n,k)\right]}{(k-\rho)(1-\mu)}\right]^{\frac{1}{k-1}}. \tag{2.3}$$

The result is sharp for the function $f(z)$ defined by (2.2).

Proof

To find the radius of starlike of order α, it sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\rho. \tag{2.4}$$

By simple calculations, we get

$$\left|\frac{zf'(z)}{f(z)}-1\right|=\left|\frac{zf'(z)-f(z)}{f(z)}\right|<\frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k|z|^{k-1}}. \tag{2.5}$$

Thus equation (2.4) satisfies if

$$\sum_{k=2}^{\infty}\left(\frac{k-\rho}{1-\rho}\right)a_k|z|^{k-1}<1. \tag{2.6}$$

Since $f(z)\in S^n_{\lambda,\alpha}(\mu)$, Lemma 2.1 conforms that

$$\sum_{k=2}^{\infty}\frac{(k-\mu)\left[\alpha[1+(k-1)\lambda]^n+(1-\alpha)\delta(n,k)\right]}{1-\mu}a_k\le 1, \tag{2.7}$$

hence, from (2.6) and (2.7), we have

$$\left(\frac{k-\rho}{1-\rho}\right)|z|^{k-1}<\frac{(k-\mu)\left[\alpha[1+(k-1)\lambda]^n+(1-\alpha)\delta(n,k)\right]}{1-\mu}. \tag{2.8}$$

Solving (2.8) for $|z|$, we get

$$|z| < \left[\frac{(1-\rho)(k-\mu)\left[\alpha[1+(k-1)\lambda]^{n}+(1-\alpha)\delta(n,k)\right]}{(1-\mu)(k-\rho)}\right]^{\frac{1}{k-1}}. \tag{2.9}$$

Thus, the proof of Theorem 2.1 is completed.
Put $n=0$ in Theorem 2.1, we get the following corollary

Corollary 2.1:

Let the function $f(z)$ defined by (1.1) be in the class $T^{*}(\mu)$, Then $f(z)$ is starlike in $|z| < r_2(k,\rho,\mu)$, where

$$r_2(k,\rho,\mu) = \inf_{k\geq 2}\left[\frac{(1-\rho)(k-\mu)}{(k-\rho)(1-\mu)}\right]^{\frac{1}{k-1}}, \quad (k\geq 2). \tag{2.10}$$

The result is sharp for the function

$$f(z) = z - \frac{1-\mu}{(k-\mu)} z^{k}, \quad (k\geq 2). \tag{2.11}$$

Put $k=3$ in Corollary 2.1, we get

Example 2.1: Let the function

$$f(z) = z - a_2 z^2 - a_3 z^3 \tag{2.12}$$

be in the class $T^{*}(\mu)$, Then $f(z)$ is starlike in $|z| < r_3(\rho,\mu)$, where

$$r_3(\rho,\mu) = \sqrt{\frac{(1-\rho)(3-\mu)}{(3-\rho)(1-\mu)}} \tag{2.13}$$

In Figure 1, graph radius of starlike in the above example by Wolfram Alpha.

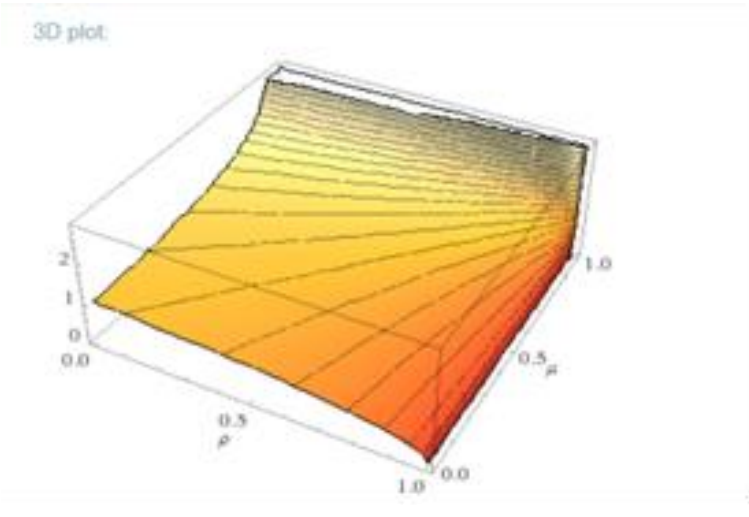

Figure-1, radius of starlike function defined by (2.12)

Put $\alpha=0$ in Theorem 2.1, we get the following corollary:

Corollary 2.2

Let the function $f(z)$ given by (1.1) be in the class $S_{\lambda}^{*}(\mu)$, then $f(z)$ is starlike in $|z|<r_4(k,n,\rho)$, where

$$r_4(k,n,\rho)=\inf_{k\geq 2}\left[\frac{(1-\rho)(k-\mu)\delta(n,k)}{(1-\mu)(k-\rho)}\right]^{\frac{1}{k-1}}, \quad (k\geq 2). \tag{2.14}$$

The result is sharp for the function

$$f(z)=z-\frac{1-\mu}{(k-\mu)\delta(n,k)}z^k, \qquad (k\geq 2; z\in U). \tag{2.15}$$

Put $k=3$ in Corollary 2.2, we get the following example:

Example 2.2: Let the function $f(z)$ defined by (2.12) be in the class $S_{\lambda}^{*}(\mu)$, then $f(z)$ is stalike in $|z|<r_5(n,\rho,\mu)$, where

$$r_5(n,\rho,\mu)=\sqrt{\frac{(1-\rho)(3-\mu)(n+1)(n+2)}{2(3-\rho)(1-\mu)}} \tag{2.16}$$

The result is sharp for the function

$$f(z)=z-\frac{2(1-\mu)}{(n+2)(n+1)(3-\mu)}z^3, \qquad (z\in U). \tag{2.17}$$

Theorem 2.2:

Let $f(z)\in S_{\lambda,\alpha}^{n}(\mu)$. Then $f(z)$ is convex of order $\rho(0\leq\rho<1)$ in $|z|<r_6(n,k,\lambda,\alpha,\mu,\rho)$, where

$$r_6\,(n,k,\lambda,\alpha,\mu)=\inf_{k\geq 2}\left[\frac{(1-\rho)(k-\mu)\left[\alpha[1+(k-1)\lambda]^n+(1-\alpha)\delta(n,k)\right]}{k(k-\rho)(1-\mu)}\right]^{\frac{1}{k-1}}. \tag{2.18}$$

The result is sharp for the function

$$f(z)=z-\frac{1-\mu}{k(k-\mu)\left[\alpha[1+(k-1)\lambda]^n+(1-\alpha)\delta(n,k)\right]}z^k, \ (z\in U, k\geq 2). \tag{2.19}$$

Proof:
By using the same technique which used in the proof of Theorem 2.1, we can show that

$$\left|\frac{zf''(z)}{f'(z)}\right|<1-\rho \quad \text{for } |z|<r_6$$

which give the assertion of Theorem 2.2.
Put $n=0$ in Theorem 2.2, we get the following corollary

Corollary 2.3:

Let the function $f(z)$ given by (1.2) be in the class $T^{*}(\mu)$, then $f(z)$ is convex of order $\rho(0\leq\rho<1)$ in $|z|<r_7(k,\rho,\mu)$, where

$$r_7(k,\rho,\mu)=\inf_{k\geq 2}\left[\frac{(1-\rho)(k-\mu)}{k(k-\rho)(1-\mu)}\right]^{\frac{1}{k-1}}, \quad (k\geq 2). \tag{2.20}$$

The result is sharp for the function

$$f(z)=z-\frac{1-\mu}{k(k-\mu)}z^k, \quad (k\geq 2, z\in U). \tag{2.21}$$

Put $k=2$ and $\mu=0$ in Corollary 2.3, we get

Example 2.3: Let the function defined by (2.12) be in the class T^*, then $f(z)$ is convex in $|z|<r_8(\rho)$, where

$$r_8(\rho)=\frac{2(1-\rho)}{2(2-\rho)} \tag{2.22}$$

The result is sharp for the function

$$f(z)=z-\frac{1}{4}z^2, \quad (z\in U). \tag{2.23}$$

In figure 2 , graph the sharp function in Example 2.3 by Complex Tool program

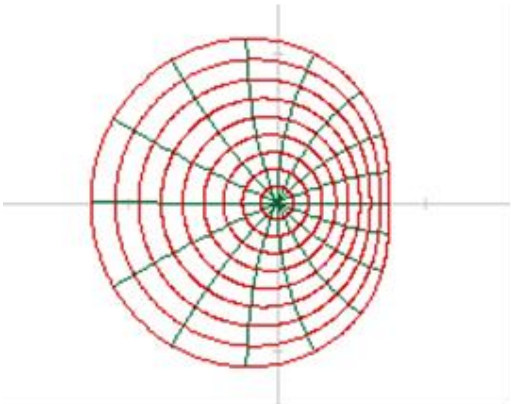

Figure 2: the image of unit disc under the function (2.23)

Put $k=3$ in Corollary 2.3, we get

Example 2.4: Let the function defined by (3.12) be in the class $T^*(\mu)$, then $f(z)$ is convex in $|z|<r_9(\rho,\mu)$, where

$$r_9(\rho,\mu)=\sqrt{\frac{(1-\rho)(3-\mu)}{3(3-\rho)(1-\mu)}} \tag{2.24}$$

The result is sharp for the function

$$f(z)=z-\frac{1-\mu}{3(3-\mu)}z^3, \quad (z\in U). \tag{2.25}$$

In Figure 3, graph the radius of convex in the above Example by Wolfram Alfa program, we get

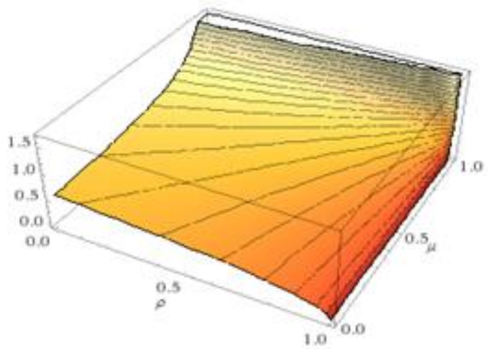

Figure 3: radius of convex function defined by (2.25)

Put $\rho=0$ in Corollary 2.3, we get the following corollary

Corollary 2.4

Let the function $f(z)$ given by (1.1) be in the class $T^*(\mu)$, Then $f(z)$ is convex in $|z|<r_9(k,\mu)$,

$$r_9(k,\mu)=\inf_{k\geq 2}\left[\frac{(k-\mu)}{k^2(1-\mu)}\right]^{\frac{1}{k-1}} . \tag{2.26}$$

The result is sharp for the function $f(z)$ given by (2.21).

RESULT

The result in Corollary 2.4 given the known result of Silverman [7, Theorem 8]

CONCLUSION

This work is a generalization for well-known radius problem of univalent functions and gave some examples.

REFERENCES

1. Ahuja O. P., Hadamard products of analytic functions defined by Ruscheweyh derivatives, in: Current topics in analytic function theory, 13{28, (H. M. Srivastava, S Owa, editors), World Sci. Publishing, Singapore,1992. http://www.researchgate.net/publication/233841957_A_class_of_multivalent_functions_with_negative_coefficients_defined_by_convolution
2. AL-Oboudi F. M., On univalent functions defined by a generalize Sâlâgean operator, Internat. J. Math.Math. Sci 27 (2004), 1429- 1436. http://dx.doi.org/10.1155/S0161171204108090
3. Lupas A. Alb, On special differential subordinations using a generalized Sâlâgean operator and Ruscheweyh derivative, Journal of Computational Analysis and Applications, 13(2011), No.1, 98-107. http://files.ele-math.com/abstracts/mia-12-61- abs.pdf]
4. Lupas A. Alb and Andrei L., New classes containing generalized Salagean operator and Ruscheweyh derivative, Acta Universitatis Apulensis. (2014), No. 38, 319 – 328. www.kurims.kyotou.ac.jp/EMIS/journals/AUA/pdf/63_1063_alina60aua.pdf]
5. Ruscheweyh S., New criteria for univalent functions, Proc. Am. Math. Soc. 49(1975), 109-115.
6. Sâlâgean G. S., Subclasses of univalent functions. Lecture Notes in Math., 1013(1983), 362-372. Springer-Verlag, Berlin Heidelberg & New York. http://www.springerlink.com/content/w753816720375137]
7. Silverman H., Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51 (1975), 109-116. http://www.ams.org/journals/proc/1975-051-01/S0002-9939-1975-0369678-0/]