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On Hilbert Space Operator Deformation Analysis in Application to Some Elements of Elasticity Theory

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ABSTRACT

The application of abstract results on contraction and extension in the Hilbert space through the concept of spectra in this paper, aims at an analytical survey of some deformation problems in elasticity theory. Results used in achieving this target were fully outlined in sections one and two while the target was realized in the last section.

Key words: Contraction, deformation, extension, Hilbert space, operators, spectra
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INTRODUCTION

In this work,[1] we make use of the normal, particularly self-adjoint and unitary operators in Hilbert space. However, since less is known of the structure of normal operators for a lack of satisfactory generalization, we, therefore, resort to finding relations that will reduce the problem of dealing with general linear operators to a more workable particular case of normal operators of which its simplest types are

$T = A + iB$,

Where the bounded linear operator T in the Hilbert space $\mathfrak{H}$ is represented by the two self-adjoint operators

$$A = ReT = \frac{1}{2}\left(T + T^*\right),\ B = \mathrm{Im}T - \frac{1}{2i}\left(T - T^*\right)$$

And

$T = VR$

Where the bounded linear isometric operator (which, in certain cases, can be chosen to be unitary, in particular if T is a one-to-one operator of the space $\mathfrak{H}$ onto itself). The applicability of these relations is restricted by the fact that neither A and B nor V and R are in general permutable, and there is no simple relation among the corresponding representations of the iterated operators $T, T^2, \ldots$
In the sequel,[2] we shall deal with other relations which are connected with extensions of a given operator. But contrary to what we usually do, we shall also allow extensions which extend beyond the given space. So by an extension of a linear operator T of Hilbert space $\mathfrak{H}$, we shall understand a linear operator T in a Hilbert space H which contains $\mathfrak{H}$ as a (not necessarily proper) subspace, such that $D_{\boldsymbol{T}} \supseteq D_T$ and $Tf = Tf$ for $f \in D_T$. We shall retain notation, $T \supset T$ which we used for ordinary extensions (where $H = \mathfrak{H}$).

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The orthogonal projection of the extension space H onto its subspace $\mathfrak{H}$ will be denoted by $P_{\mathfrak{H}}$ or simply by P. Among the extensions of a bounded linear operator T in $\mathfrak{H}$ (with $D_T = \mathfrak{H}$), we shall consider in particular those which are of the form PS where S is bounded linear transformation of an extension space H. We express this relation

$$T \subseteq PS$$

by saying that T is the projection of the operator onto $\mathfrak{H}$, in symbols

$$T = pr_{\mathfrak{H}}S \text{ or simply } T = prS \tag{1.1}$$

It is obvious that the relations $T_i = prS_i (i = 1, 2)$ imply the relation

$$a_1T_1 + a_2T_2 = pr\left(a_1S_1 + a_2S_2\right) \tag{1.2}$$

(of course, S_1 and S_2 are operator in the same extension space H). Relation (1) also implies that

$$T^* = prS^* \tag{1.3}$$

Finally, the uniform, strong, or weak convergence of a sequence $\{S_n\}$ implies convergence of the same type for the sequence $\{T_n\}$ where $T_n = prS_n$. If H and H' are two extension spaces of the same space $\mathfrak{H}, S, S'$ are bounded linear operator of H and H', respectively, then we shall say that the structures $\{H, S, \mathfrak{H}\}$ and $\{H', S', \mathfrak{H}\}$ are isomorphic if H can be mapped isometrically onto H' in such a way that the elements of the common subspace $\mathfrak{H}$ are left invariant and that $f \to f'$ implies $Sf \to S'f'$. If $\{S\omega\}_{\omega\in\Omega}$ and $\{S'_\omega\}_{\omega\in\Omega}$ are two families of bounded linear transformations in H and H', respectively, we define the isomorphism of the structures.

The terminology T is the compression of S in $\mathfrak{H}$, and S is the dilation of T to H according to the proof by HALMOS.

In fact, we have[3]

$$\left(T_0, T^*g\right)\left(TT_0, g\right)\left(PSPT_0, g\right) = \left(T_0, PS^*Pg\right) = \left(T_0, PS^*g\right) \text{ for } T_0, g \in \mathfrak{H}$$

and

$$\| T_n - T_m \| \leqq \| S_n - S_m \|, \| \left(T_n - T_m\right)T_0 \| \leqq \| \left(S_n - S_m\right)T_0 \| \text{ for } T_0 \in \mathfrak{H}$$

and

$$\left(\left(T_n - T_m\right)T_0, g\right) = \left(\left(S_n - S_m\right)T_0, g\right) \text{ for } T_0, g \in \mathfrak{H}.$$

$\left(H.S_\omega, \mathfrak{H}\right)_{\omega\in\Omega}$ and $\{H', S'_\omega, \mathfrak{H}\}_{\omega\in\Omega}$ in the same manner by requiring that $T_0 \to T^*$ imply $S_\omega T_0 \to S'_\omega T^*$ for all $\omega \in \Omega$.

It is obvious that, from the point of view of extensions of operators in $\mathfrak{H}$ which extend beyond $\mathfrak{H}$, two extensions which give rise to two isomorphic structures can be considered as identical. In the sequel, when speaking of Hilbert spaces, we shall mean both real and complex spaces. If we wish to distinguish between real and complex spaces, we shall say so explicitly. Of course, an extension space H of $\mathfrak{H}$ is always of the same type (real or complex) as $\mathfrak{H}$.

Theorem 1.1:[4,5] A necessary and sufficient condition for the operator g given in a set E of the space C to be extendable to the entire space C so as to define there a linear operator of norm $\leq M$ is that

$$\| \sum_{k=1}^{n} c_k GT_0 \| \leq M \| \sum_{k=1}^{n} c_k T_0 \| \tag{1.4}$$

for every linear combination of remnants of E.

Proof:

Consider the inequality (1.4), we start the proof as below by simplifying (1.4). Set $M=1$, this can be done without loss of generality. Continue by defining the functional G for G_* which can be represented as linear combinations of elements of X. If

$$G_* = \sum_{i=1}^{n} c_k T_0 ,$$

Set $GG_* \sum_{i=1}^{n} c_k GT_0$;
We show that this definition is single valued, meaning that if

$$\sum_{i=1}^{n} c_k T_0 = \sum_{i=1}^{n} c_k' T_0, \text{ then } \sum_{i=1}^{n} c_k GT_0 = \sum c_k' GT_0$$

Taking differences, we have

$$\sum_{i=1}^{n} c_k'' T_0 = 0 \Rightarrow \sum_{i=1}^{n} c_k'' GT_0 \left(c_k'' = c_k - c_k'\right)$$

This follows immediately from the condition (1.1). Hence, we observe that these linear combinations of g_* form a linear manifold X' and the operator g extended to X', is obviously additive, homogeneous and from (1.1) is also bounded by $M=1$ having established the above we adjourn new elements for observations.
Denote by $G_{*(1)}$ and $G_{*(2)}$ two arbitrary elements of X' and by $GG_{*(1)}$ and $GG_{*(2)}$ the corresponding values of the extension of G to X'. Let T_0 be an element of the space C which does not belong to X'.

Since $GG_{*(1)} - GG_{*(2)} = G\left(G_{*(1)} - G_{*(2)}\right) \leq \| G_{*(1)} - G_{*(2)} \| \leq \| G_{*(1)} - T_0 \| + \| G_{*(2)} - T_0 \|$

Hence,

$$GG_{*(1)} - \| G_{*(1)} - T_0 \| \leq GG_{*(1)} + \| G_{*(1)} \|$$

So if we vary g in such a way that it runs through all elements of X', the quantities

$$GG_* - \| T_0 - G_* \| \text{ and } GG_* + \| T_0 - g_* \|$$

Form for fixed T_0, two classes of real numbers, the first of which lies to the left of the second, and consequently with one or more numbers included between the two classes. Set GT_0 equal to one of these values, then for every element g of X'

$$GG_* - \| T_0 - G_* \| \leq GT_0 \leq GG_* + \| T_0 - G_* \|$$

and replacing G_* by $-G_*$ (since X' is a linear manifold), we have

$$\| GT_0 + GG_* \| \leq \| T_0 + G_* \| \tag{1.5}$$

Extending the operator g to linear combination of T_0 and element g_* of X

$$G\left(cT_0 + G_*\right) = cGT_0 + GT_0$$

and we shall have

$$\left|G\left(cT_0 + G_*\right)\right| \leq cT_0 + G_*$$

Placing $c=0$ makes the above to be nothing other than condition (1.1) and $c \neq 0$ in (1.1) and replacing G_* by $\frac{1}{c} G_*$ still in same (1.2) gives

$$g\left(cT_0 - G_*\right) = \left|cGT_0 + GG_*\right| = \left|c\right|\left|GT_0 + G\frac{G_*}{c}\right| \le \left|c\right|\left|T_0 + \frac{G_*}{c}\right|$$

Therefore, condition (1.4) is again fulfilled in the linear set $X^{''}$ formed by linear operator of T_0 and elements G_* of $X^{'}$ remaining additive, homogeneous and bounded consequent on the above, if for T_0, we choose a successive sequence of operators, $1, x, x^2, \ldots,$ say whose linear combinations and themselves or with elements of $X^{'}$ are everywhere dense in C in the sense of uniform convergence the operator G will be defined successively for this everywhere dense set without it being necessary to change the bound $M = 1$. Immediately, we pass the limit to extend G to the entire space, we discover that the following theorem has been proved.

This theorem also holds for the space of complex continuous operators. The necessity of condition (1.4) with complex coefficients c_k, is evident; we shall show how the proof of the sufficiency can be reduced to the real case.

We first extend the operator G to the linear set E formed by complex linear combinations of elements of E, just as was done in the real case. On E, G will be homogeneous even with respect to a complex numerical factor, therefore, in particular, we shall have $G\left(ih\right) = iGh$ for all elements H of E. Denote by $G_1 h$ the real part of GH; inequality (4) holds for G_1 in place of g, if we consider only linear combinations of elements of E with real coefficients c_k. Then, we extend this real-valued functional G_1 to the entire space $C_{complex}$, be real-valued, additive, homogenous with respect to real factors, and bounded by M. We shall show that

$$BT_0 = G_1 T_0 - iG_1(iT_0)$$

Furnishes the desired extension of the operator G. To accomplish this it is necessary to show that the operator B is additive, homogenous with respect to complex factors, bounded by M, and finally, that it coincides on X with the operator G. Additivity follows immediately from the additivity of G_1. A little more calculation is necessary to establish homogeneity with respect to complex factor $c = a + ib$:

$$\begin{aligned} B\left(cT_0\right) &= G_1\left(aT_0 + biT_0\right) - iG_1\left(aiT_0 - bT_0\right) \\ &= aG_1T_0 + bG_1\left(iT_0\right) - iaG_1\left(iT_0\right) + ibG_1T_0 \\ &= \left(a + ib\right)\left(G_1T_0 - iG_1\left(iT_0\right)\right) = cBT_0; \end{aligned}$$

In this calculation, we have made use of the additivity of G_1 and of its homogeneity with respect to real factors. To show that B is bounded by M, we set

$$BT_0 = re^{it}\,(r \ge 0)$$

For arbitrary fixed T; then we shall have

$$\left|BT_0\right| = e^{-it}BT_0 = B\left(e^{-it}T_0\right) = G_1\left(e^{-it}T_0\right) \le M \,\|\, e^{-it}T_0 \,\|\, = M \,\|\, T_0 \,\|$$

the third equation is motivated by the fact that $B(e^{-it}T_0)$, being equal to r, is real-valued. Finally, to show that $BH = GH$ for every H of X, we note that $-G_1(ig)$ is by definition equal to the real part of $-G\left(ig\right) = -iAg$, hence equal to the imaginary part of GH and that consequently

$$iGH = \left(G_1H - iG_1\left(iH\right)\right)$$

Which completes the proof.

Generalized spectral families and the NEUMARK'S theorem

In extensions which extend beyond the given space M.A NEUMARK; he centers on self-adjoint extensions of symmetric operators in particular and consequently, if S is a symmetric operator in the complex Hilbert space $\mathfrak{H}$ (with D_s dense in $\mathfrak{H}$), we know that S cannot be extended to a self-adjoint operator without extending beyond $\mathfrak{H}$ except when the deficiency indices m and n of S are equal. On the other hand, there always exists self-adjoint extensions of S if one allows these extensions to extend beyond the space $\mathfrak{H}$.
This is easily proved: Choose, in a Hilbert space $\mathfrak{H}^{'}$, a symmetric operator S' but in reverse order. One can take, for example, $\mathfrak{H}^{'} = \mathfrak{H}$ and $S^{'} = -S$. Having done this, we consider the product space $H = \mathfrak{H} \times \mathfrak{H}^{'}$ whose elements are pairs $\left\{T, T^{'}\right\}\left(T \in \mathfrak{H}, T^{'} \in \mathfrak{H}\right)$ and in which the vector operations and metrics are defined as follows:

$$c\left\{T, T^{'}\right\} = \left\{cT, cT^{'}\right\}; \left\{T_1 T_1^{'}\right\} + \left\{T_2, T_2^{'}\right\} = \left\{T_1 + T_2, T_1^{'} + T_2^{'}\right\};$$

$$\left(\left\{T_1, T_1^{'}\right\}, \left\{T_2, T_2^{'}\right\}\right) = \left(T_1, T_2\right) + \left(T_1^{'}, T_2^{'}\right).$$

If we identify the element T in $\mathfrak{H}$ with the element $\left\{T, 0\right\}$ in H, we embed $\mathfrak{H}$ in H as a subspace of the latter. The operator

$$S\left\{T, T^{'}\right\} = \left\{ST, S^{'}T^{'}\right\}\left(T \in D_s, T^{'} \in D_s\right)$$

is then, as can easily be seen, a symmetric operator in H having deficiency indices $m+n, n+m$. Consequently, S can be extended, without extending beyond H, to a self-adjoint operator A in H. Since we have

$$S \subseteqq S \subseteqq A$$

(where the first extension is obtained by extension from $\mathfrak{H}$ to H), we obtain a self-adjoint extension A of S. Let,[4]

$$A\int_{-\infty}^{\infty} \lambda dE_\lambda$$

be the spectral decomposition of A. We have the relations

$$\left(ST, g\right) = \left(AT, Pg\right) = \int_{-\infty}^{\infty} \lambda d\left(E_\lambda T, Pg\right) = \int_{-\infty}^{\infty} \lambda d\left(PE_\lambda T, g\right),$$

$$\left\|ST\right\|^2 = \left\|AT\right\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\left(E_\lambda T, T\right) = \int_{-\infty}^{\infty} \lambda^2 d\left(PE_\lambda T, T\right).$$

For $T \in D_s, g \in \mathfrak{H}$. Setting

$$B_\lambda = prE_\lambda \tag{1.2.1}$$

We obtain a family $\left\{B_\lambda\right\}_{-\infty<\lambda<\infty}$ of bounded self-adjoint operators in the space $\mathfrak{H}$, which have the following properties:
a. $B_\lambda \leqq B_\mu$ for $\lambda < \mu$;
b. $B_{\lambda+0} = B_\lambda$;
c. $B_\lambda \to 0$ as $\lambda \to -\infty$; $B_\lambda \to I$ as $\lambda \to +\infty$.
Every one-parameter family of the bounded self-adjoint operator has these properties will be called a generalized spectral family. If this family consists of projections (which are then, as a consequence of a), mutually permutable, then we have an ordinary spectral family.

According to what we just proved, we can assign to each symmetric operator S in $\mathfrak{H}$ a generalized spectral family $\{B_\lambda\}$ in such a way that the equations

$$\left(ST,g\right)=\int_{-\infty}^{\infty}\lambda d\left(B_\lambda T,g\right),\|ST\|^2=\int_{-\infty}^{\infty}\lambda^2 d\left(B_\lambda T,T\right) \tag{1.2.2}$$

Are satisfied for $T\in D_s$, $g\in\mathfrak{H}$ (where the integral in the second equation can also converge for certain T which do not belong to D_s).

Theorem 1.2.1:[6] Every generalized spectral family $\{B_\lambda\}$ can be represented in the form (4), as the projection of an ordinary spectral family $\{E_\lambda\}$. One can even require the extension space H to be minimal in the sense that it be spanned by the elements of the form $E_\lambda T$ where $T\in\mathfrak{H},-\infty<\lambda<\infty$; in this case, the structure $\{H,E_\lambda,\mathfrak{H}\}_{-\infty<\lambda<\infty}$ is determined to within an isomorphism. We shall prove this theorem later as a corollary to the principal theorem.

Proof of 1.2 (NEUMARK'S)

Let $\{B_\lambda\}_{-\infty<\lambda<\infty}$ be a generalized spectral family in $\mathfrak{H}$. Set $B_\infty=\lim_{\lambda\to\infty}B_\lambda=I$ and $B_{-\infty}=\lim_{\lambda\to\infty}B_\lambda=0$. We assign the operator

$$B_\Delta=B_b-B_a$$

To each half-open interval

$$\Delta=\left(a,b\right]\qquad(\text{where}-\infty\leqq a<b\leqq\infty$$

and the operator

$$B\omega=\sum_i B_{\Delta_i};$$

to each set ω which consists of a finite number of disjoint intervals Δ_i; this definition obviously does not depend on the particular choice of the decomposition of ω. For $\Omega=\left(-\infty,\infty\right]$ we have $B_\Omega=I$, and for the void set Θ we have $B_\Theta=0$. The family K of these sets ω, including Ω and Θ, is clearly closed with respect to subtraction of any two sets and with respect to the operation of forming unions and intersections of a finite number of sets. B_ω is a positive additive set function defined on K; more precisely, B_ω is for all $\omega\in K$, a self-adjoint operator such that

$$0\leqq B\leqq I,B_\Theta=0,\ B_\Omega=I,B_{\omega_1\cup\omega_2}=B_{\omega_1}+B_{\omega_{12}},\text{where}\,\omega_1\cap\omega_2=\Theta$$

We shall also consider K as a $*$-semi-group; we do this by setting

$$\omega_1\omega_2=\omega_1\cap\omega_2,\omega^*=\omega,\mu=\Omega$$

We shall see that B_ω, considered as a function defined on this $*$-semi-group, satisfies the conditions of the principal theorem.

Condition (a) is satisfied in an obvious manner, condition (b) means that

$$s=\sum_i\sum_j\left(B_{\omega_i\cap\omega_j}g_j,g_i\right)\geqq 0$$

For arbitrary $\omega_1,\dots,\omega_n\in K$ and $g_1,\dots,g_n\in\mathfrak{H}$. To prove this inequality, we first consider the intersections

$$\pi=\omega_1\pm\cap\ \omega_2\pm\cap\dots\cap\ \omega_n\pm\qquad\left(\in K\right)$$

Where each time we can choose one of the signs $+$ or $-$ in an arbitrary manner; ω^+ denotes the set ω itself and ω^- its complement $\Omega-\omega$. Two intersections $\neq$ corresponding to different variations of the sign are obviously disjoint. Each set $\omega_i\cap\omega_j\left(i\overset{\leq}{<}j\right)$ is the union of certain of these $\neq$, namely, of all those obtained by choosing the sign $+$ for i and j, that is, of all those which are contained in $\omega_i\cap\omega_j$. In virtue of the additivity of B_ω as a set operator, the sum s then decomposes into a sum of terms of the form

$$\left(B_\pi g_j, g_i\right)$$

We combine the terms corresponding to the same $\neq$ into a partial sum $s_{\neq}$; the latter extends to all the pairs of indices (i,j) for which $\omega_i \cap \omega_j \supseteqq \pi$, that is for which $\omega_i \supseteqq \pi$ and $\omega_j \supseteqq \pi$ simultaneously. Suppose $i_1, i_2, \ldots, i_r$ are those values of the index i for which ω_i contains the fixed set π; we then have

$$s_\pi = \sum_{h=1}^{r} \sum_{k=1}^{r} \left(B_\pi g_{i_k}, g_{i_h}\right) = \left(B_\pi g, g\right) \text{with } g = \sum_{h=1}^{r} g_{i_h},$$

and consequently, $s_\pi \geqq 0$. Since this is true for all the π, it follows that $s = \sum_\pi s_\pi \geqq 0$, which was to be proved.

Let us now pass on to condition (c). Suppose ω is a fixed element in K and set

$$\omega_i^{'} = \omega_i \cap \omega^+, \omega_i^{''} = \omega_i \cap \omega^- \left(i = 1, 2, \ldots, n\right)$$

Applying the inequality $s \geqq 0$, which we have just proved, to $\omega_i^{'}$ and $\omega_i^{''}$ instead of to ω_i, we obtain

$$s^{'} = \sum_i \sum_j \left(B_{\omega_i^{'} \cap \omega_j^{'}} g_j, g_i\right) \geqq 0,\ s^{''} = \sum_i \sum_j \left(B_{\omega_i^{''} \cap \omega_j^{''}} g_j, g_i\right) \geqq 0.$$

Since $\omega_i^{'} \cap \omega_j^{'}$ and $\omega_i^{''} \cap \omega_j^{''}$ are to be contained in the disjoint sets ω^+, ω^-, they are also disjoint; since their union is equal to $\omega_i \cap \omega_j$, it follows from the additivity of B_ω that $s^{'} + s^{''} = s$. Consequently, we have $0 \leqq s^{'} \leqq s$, that is

$$0 \leqq \sum_i \sum_j \left(B_{\omega_i \cap \omega \cap \omega \cap \omega_j} g_j, g_i\right) \leqq \sum_i \sum_j \left(B_{\omega_i \cap \omega_j} g_j, g_i\right),$$

and hence condition (c) is satisfied with $C_\omega = 1$.

We[7] can then apply the principal theorem of section 1.4.2. Hence, there exists a representation $\{E_\omega\}$ of the $*$-semi-group K on a minimal extension space H such that

$$B_\omega = prE_\omega;$$

here, "minimal" means that the space H is spanned by the elements of the form $E_\omega f$ where $f \in \mathfrak{H}, \omega \in K.$ It follows from the structure of K as a $*$-semi-group that E_ω is a projection, $E_\Omega = I$, and

$$E_{\omega_1 \cap \omega_2} = E_{\omega_1} E_{\omega_2} \tag{1.2.3}$$

We also have

$$E_{\omega_1 \cup \omega_2} = E_{\omega_1} + E_{\omega_2},\ \text{when}\ \omega_1 \cap \omega_2 = \Theta; \tag{1.2.4}$$

This follows, virtue of the fact that H is minimal, from the fact that, B_ω being an additive operator of ω, we have

$$B_{(\omega_1 \cup \omega_2 \cap \omega)} = B_{\omega_1 \cap \omega} + B_{\omega_2 \cap \omega}$$

for all $\omega \in K$. In particular, we have $E_\Theta = E_{\Theta \cup \Theta} = E_\Theta + E_\Theta$, and hence $E_\Theta = 0$.

We set

$$E_\lambda = E_{(-\infty, \lambda]} \text{ for } -\infty < \lambda < \infty;$$

Since $B_{(-\infty,\lambda]} = B_\lambda - B_{-\infty} = B_\lambda$, we then have

$$B_{\lambda} = prE_{\lambda}, \tag{1.2.5}$$

And by (27) or (28),

$E_{\lambda} \leqq E_{\mu}$ for $\lambda < \mu$

We finally arrive at the relations

$$E_{\lambda} \to E_{\mu} \quad \text{as } \lambda \to \mu + 0;$$

$$E_{\lambda} \to E_{\Theta} = 0 \quad \text{as } \lambda \to -\infty;$$

$$E_{\lambda} \to E_{\Omega} = I \quad \text{as } \lambda \to \infty;$$

Which are consequences, in virtue of the fact that H is minimal, of the relations

$$B_{(-\infty,\lambda]\cap\omega} \to B_{(-\infty,\mu]\cap\omega} \quad \text{as } \lambda \to \mu + 0,$$

$$B_{(-\infty,\lambda]\cap\omega} \to 0 = B_{\Theta\cap\omega} \quad \text{as } \lambda \to -\infty,$$

$$B_{(-\infty,\lambda]\cap\omega} \to B_{\omega} = B_{\Omega\cap\omega} \quad \text{as } \lambda \to +\infty$$

Which are valid for all fixed ω.
Hence, $\{E_{\omega}\}$ is an ordinary spectral family. Since each of the E_{ω} is derived from the E_{λ} by forming differences and sums or passing to the limit $(\lambda \to \pm\infty)$, the space H is also minimal with respect to $\{E_{\lambda}\}$, and the structure $\{H, E_{\lambda}, \mathfrak{H}\}$ is determined to within an isomorphism. This completes the proof of theorem 1.2.1.
We observe that[8] if H is minimal, every interval of the constancy of B_{λ} is also an interval of constancy for E_{λ}. In fact, if $a \leqq \lambda < b$ is an interval of the constancy of B_{λ}, we have

$$\|\left(E_{\lambda} - E_{a}\right)E_{\mu}T_{0}\|^{2} = \|\left(E_{\min\{\lambda,\mu\}} - E_{\min\{a,\mu\}}\right)T_{0}\|^{2} = \left(\left(E_{\min\{\lambda,\mu\}} - E_{\min\{a,\mu\}}\right)T_{0}, T_{0}\right)$$

$$= \left(P\left(E_{\min\{\lambda,\mu\}} - E_{\min\{a,\mu\}}\right)T_{0}, T_{0}\right) = \left(\left(B_{\min\{\lambda,\mu\}} - B_{\min\{a,\mu\}}\right)T_{0}, T_{0}\right) = 0$$

for $T_{0} \in \mathfrak{H}, a \leqq \lambda < b$, and $\propto$ an arbitrary real number; hence

$$\left(E_{\lambda} - E_{a}\right)g = 0$$

for every element g of the form $E_{\mu}T_{0}\left(T_{0} \in \mathfrak{H}\right)$. Since these elements g span the space H, we have $E_{\lambda} - E_{a} = 0, E_{\lambda} = E_{a}$, which completes the proof of the theorem. The simplest case of the Neumark theorem occurs when the family $\{B_{\lambda}\}$ is generated by a self-adjoint operator A such that $0 \leqq A \leqq I$, in the following manner:

$B_{\lambda} = 0$ for $\lambda < a$, $B_{\lambda} = A$ for $a \leqq \lambda < b, B_{\lambda} = I$ for $\lambda \geqq b$.

We thus obtain the following corollary.
Corollary 1.2.2:[9] Every self-adjoint transformation A in the Hilbert space $\mathfrak{H}$, such that $0 \leqq A \leqq I$, can be represented in the form

$$A = prQ$$

where Q is a projection in an extension space H. In brief: A is the projection of a projection.
This corollary can also be proved directly without recourse to the general Neumark theorem. The following construction is due to E.A. Michael.
Consider the product space $H = \mathfrak{H} \times \mathfrak{H}$; by identifying the element T in $\mathfrak{H}$ with the element $\{T, 0\}$ in H, we embed $\mathfrak{H}$ in H as a subspace of the latter. If we write the elements H as one-column matrices $\begin{pmatrix} T \\ g \end{pmatrix}$, then every bounded linear operator T in H can be represented in the form of a matrix

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \tag{1.2.6}$$

Whose elements T_{ik} bounded linear operator in $\mathfrak{H}$. It is easily verified that the matrix addition and multiplication of the corresponding matrices correspond to the addition and multiplication of the operators. Moreover, relation (6) implies that

$$T^* = \begin{pmatrix} T_{11}^* & T_{12}^* \\ T_{21}^* & T_{22}^* \end{pmatrix}$$

Finally, we have

$$T = prT$$

If and only if

$$T_n = T$$

This done, we consider the operator

$$Q = \begin{pmatrix} A & B \\ B & I - A \end{pmatrix} \text{ with } B = \left[A(I - A)\right]^{\frac{1}{2}}$$

It is clear that Q is self-adjoint and that $A = prQ$. It remains only to show that $Q^2 = Q$, which is easily done by calculating the square of the matrix Q. The following theorem is another, less special, consequence of the Neumark theorem.
Theorem 1.2.3:[10] Every finite or infinite sequence $\{A_n\}$ of bounded self-adjoint operators in the Hilbert space $\mathfrak{H}$ such that

$$A_n \geqq 0, \sum A_n = 1$$

can be represented in the form

$$A_n = prQ_n \left(n = 1, 2, \ldots\right),$$

Where $\{Q_n\}$ is sequence of projections of an extension space H for which

$$Q_n Q_m = 0\left(m \neq n\right), \sum Q_n = I$$

In fact, one has only to apply the Neumark theorem to the generalized spectral family $\{B_\lambda\}$ defined by

$$B_\lambda = \sum_{n \leqq \lambda} A_n$$

If $\{E_\lambda\}$ is an ordinary spectral family in a minimal extension space such that $B_\lambda = prE_\lambda$, the function E_λ of λ increases only at the points n where it has jumps

$Q_n = E_n - E_{n-0}$;

these operators Q_n satisfy the requirements of the theorem. This theorem, in its turn, has the following theorem as a consequence.

Theorem 1.2.4:[11] Every finite or infinite sequence $\{T_n\}$ of bounded linear operators in the complex Hilbert space $\mathfrak{H}$ can be represented by means of a sequence $\{N_n\}$ of bounded normal operators in an extension space H in the form

$$T_n = prN_n\,(n = 1, 2, \ldots),$$

Where the N_n is pairwise doubly permutable. If any of the operator T_n is self-adjoint, the corresponding N_n can also be chosen to be self-adjoint. We first consider the case where all the operators T_n are self-adjoint. If m_n and M_n are the greatest lower and least upper bounds of T_n, we set

$$A_n = \frac{1}{2^n\left(M_n - m_n + 1\right)}\left(T_n - m_n I\right) \qquad (n = 1, 2, \ldots);$$

Then, we obviously have

$$A_n \geqq 0, \sum\nolimits_n A_n \leqq I$$

If we again set

$$A = I - \sum\nolimits_n A_n$$

we obtain a sequence $A, A_1, A_2, \ldots,$ of operators which satisfies the hypotheses of the preceding theorem and which consequently can be represented in the form

$$A_n = prQ_n \qquad (n = 1, 2, \ldots)$$

In terms of the projections Q_n, which are pairwise orthogonal (and consequently permutable). It follows that

$$T_n = prS_n\ (n = 1, 2, \ldots)$$

with

$$S_n = m_n I + 2^n\left(M_n - m_n + 1\right)Q_n,$$

where the operators S_n are self-adjoint and mutually permutable. The general case is reducible to the particular case of self-adjoint operators by replacing each operator T_n in the given sequence by the two self-adjoint operators $\operatorname{Re}T_n$ and $\operatorname{Im}T_n$. In fact, since the representation

$$\operatorname{Re}T_n = prS_{2n},\ \operatorname{Im}T_n = prS_{2n+1} \quad (n = 1, 2, \ldots)$$

is possible by means of bounded self-adjoint pairwise permutable operators S_1, the representation

$$T_n = prN_n \quad (n = 1, 2, \ldots)$$

follows from this by means of the normal pairwise doubly permutable operators $N_n = S_{2n} + iS_{2n+1}$. For a self-adjoint T_n, we have $T_n = \operatorname{Re}T_n$, and we can then choose $S_{2n+1} = 0$ and hence $N_n = S_{2n}$.

Sequences of moments

1. The following theorem is closely related to the theorem on extension

Theorem 1.3.1:[12] Suppose $\{A_n\}(n=0,1,\ldots)$ is a sequence of bounded self-adjoint operators in the Hilbert space $\mathfrak{H}$ satisfying the following conditions

$$(a_M)\begin{cases} \text{for every polynomial } a_0+a_1\lambda+a_2\lambda^2+\ldots+a_n\lambda^n \\ \text{with real coefficients which assumes non}-\text{negative values in the interval} \\ -M\leqq\lambda\leqq M, \text{we have } a_0A_0+a_1A_1+a_2A_2+\ldots+a_nA_n\geqq 0 \end{cases}$$

$$A_0=I. \tag{1.3.1}$$

Then, there exists a self-adjoint operator A in an extension space H such that

$$A_n=prA^n \quad (n=0,1,\ldots,) \tag{1.3.2}$$

The proof of this theorem is based on the concept of the principal theorem, as in subsection 1.4.2 below. Suppose Γ is the $*$-semi-group of non-negative integers n with addition as the "semi-group operation" and with the identity operation $n^*=n$ as the "$*$-opeartion;" then the "unit" element is the number 0. Every representation of Γ is obviously of the form $\{A^n\}$ where A is a bounded self-adjoint operator. We shall show that the sequence $\{A_n\}(n=0,1,\ldots,)$ visualized in theorem 3.2, considered as a function of the variable element n in the $*$-semi-group Γ, satisfies the conditions of the principal theorem. Condition (a) is obviously satisfied; as for the other two conditions, one proves them using the integral formula

$$A_n=\int_{-M-o}^{M}\lambda^n dB_\lambda$$

established in sec. 3, where $\{B_\lambda\}$ is a generalized spectral family on the interval $[-M,M]$. In fact, if $\{g_n\}(n=0,1,\ldots)$ is any sequence of elements in $\mathfrak{H}$, which are almost all equal to 0, we have

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left(A_{i+k}g_k,g_i\right)=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\int_{-M-0}^{M}\lambda^{i+k}d\left(B_\lambda g_k,g_i\right)$$

$$=\int_{-M-0}^{M}\left(B(d\lambda)g(\lambda),g(\lambda)\right)\geqq 0$$

Where we have set

$$g(\lambda)=\sum_{i=0}^{\infty}\lambda^i g_i$$

and where $B(\Delta)$ denotes the positive, additive interval function generated by B_λ, that is $B(\Delta)=B_b-B_a$ for $\Delta=(a,b]$. Furthermore, for $r=0,1,\ldots,$ we have that

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left(A_{i+2r+k}g_k,g\right)=\int_{-M-0}^{M}\lambda^{2r}(B\left(d(\lambda)g(\lambda),g(\lambda)\right)$$

$$\leqq M^{2r}\int_{-M-0}^{M}\left(B(d\lambda)g(\lambda),g(\lambda)\right)=M^{2r}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left(A_{i+k}g_k,g_i\right).$$

Thus, we see that conditions (b) and (c) are also satisfied, and one can then apply the principal theorem. Furthermore, one can require that H be minimal in the sense that it be spanned by elements of the form A^nT where $T\in\mathfrak{H}$ and $n=0,1,\ldots;$ in this case, this structure $\{H,A,\mathfrak{H}\}$ is determined to within an isomorphism, and we have

$$\|A\|\leqq M$$

We observe that if $\{B_\lambda\}$ is a generalized spectral family on the interval $[-M,M]$ (that is, $B_\lambda=0$ for $\lambda<-M$ and $B_\lambda=I$ for $\lambda\leqq M$), the operators

$$A_n = \int_{-M-0}^{M} \lambda^n dB_\lambda \quad (n = 0,1,\ldots) \tag{1.3.3}$$

satisfy conditions (a_M) and (β). Conversely, if these conditions are satisfied, the sequence $\{A_n\}$ has an integral decomposition of the form (8) with $\{B_\lambda\}$ on $[-M, M]$. This clearly follows from theorem 3.4 if we make use of the directly, in fact, the correspondence between the polynomials

$$p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_n\lambda^n$$

and the self-adjoint operators

$$A(p) = a_0 I + a_1 A_1 + a_2 A_2 + a_n A_n$$

which is homogenous, additive and of positive type with respect to the interval $-M \leqq \lambda \leqq M$, can be extended, with preservation of these properties, to a vaster class of functions which comprises among others, the discontinuous functions

$$e_\mu(\lambda) = \begin{cases} 1 \; for \; \lambda \leqq \mu \\ 0 \; for \; \lambda > \mu \end{cases}$$

and then we obtain representation (8) by setting

$$B_\mu = A(e_\mu).$$

We[13] have only to repeat verbatim the line of the argument of one of the usual proofs of the spectral decomposition of a bounded self-adjoint operator A, letting A_n play the role of A^n. The only difference is that now the difference is that now the correspondence $p(\lambda) \to p(A)$ and its extension are no longer multiplicative and that consequently the relation $e_\mu^2(\lambda) \equiv e_\mu(\lambda)$ does not imply that B_μ^2 is equal to B_μ and hence that B_μ is in general not a projection.
According to theorem 3.1, $\{B_\lambda\}$ is the projection of an ordinary spectral family $\{E_\lambda\}$, which one can choose in such a way that it is also on $[-M, M]$, and then (7) follows from (8) by setting

$$A = \int_{-M-0}^{M} \lambda dE_\lambda$$

We shall return to this theorem later and prove it as a corollary to the principal theorem.

2. If we replace condition (β) by less restrictive condition

$$A_0 \leqq I, \tag{1.3.4}$$

Then representation (7) of the sequence $\{A_n\}$ will still be possible, if only starting from $n = 1$, everything reduces to showing that if the sequence

$$\{A_0, A_1, A_2, \ldots\}$$

Satisfies conditions (a_M) and (β'), the sequence

$$\{I, A_1, A_2, \ldots,\}$$

Satisfies condition (a_M). However, if $p(\lambda) = a_0 + a_1\lambda + \ldots + a_n\lambda^n \geqq 0$ in $[-M, M]$, we have in particular that $p(0) = a_0 \geqq 0;$ since by assumption, $a_0 A_0 + a_1 A_1 + \ldots + a_n A_n \geqq 0$ and $I - A \geqq 0$, it follows that

$$a_0 I + a_1 A_1 + \ldots + a_n A_n = a_0 (I - A_0) + a_0 A_0 + a_1 A_1 + \ldots + a_n A_n \geqq 0$$

One of the most interesting consequences of the representation

$$A_n = prA^n \quad (n = 1,2,\ldots)$$

Is the following. We have

$$\left(A_2T,T_0\right)=\left(PA^2T,T_0\right)=\left(A^2T,T\right)=\left\|AT_0\right\|^2$$

$$\geqq\left\|PAT_0\right\|^2=\left(APAT_0,T_0\right)=\left(PAPAT_0,T_0\right)=(A_1^2T_0,T_0)$$

for all $T\in\mathfrak{H}$, where equality holds if and only if

$$AT=PAT_0=A_1T_0.$$

If this case occurs for all $T\in\mathfrak{H}$, we have

$$A^2T_0=A\left(AT_0\right)=A\left(A_1T_0\right)=A_1\left(A_1T_0\right)=A_1^2T_0,$$

$$A^3T_0=A\left(A^2T_0\right)=A\left(A_1^2T_0\right)=A_1\left(A_1^2T_o\right)=A_1^3T_0,\ \text{etc.}$$

And hence

$$A_nT_0=PA^nT_0=A_1^nT_0\quad (n=1,2,\ldots,)$$

We have thus obtained the following result.
If the sequence $A_0,A_1,A_2,...$ of bounded self-adjoint operators in the Hilbert space $\mathfrak{H}$ satisfies hypotheses $\left(a_M\right)$ and $\left(\beta^{'}\right)$, then the inequality

$$A_1^2\leqq A_2 \tag{1.3.5}$$

holds, where equality occurs if and only if $A_n=A_1^n\ \left(n=1,2,\ldots\right)$. Inequality (9) is due to R.V Kadison who proved it differently and used it in his researches on algebraic invariants of operator algebras. Moreover, one can also omit hypotheses $\left(\beta^{'}\right)$, and then the following inequality

$$A_1^2\leqq\left\|A_0\right\|A_2 \tag{1.3.6}$$

Is obtained; in fact, we have only to apply inequality (9) to the sequence $\left\{\left\|A_0\right\|^{-1}A_n\right\}$.

Normal extensions

We[14] proved previously in particular that every bounded linear operator T in the complex Hilbert space $\mathfrak{H}$ can be represented as the projection of a normal operator in an extension space. The question arises: Does T even have a normal extension N?
If a normal extension N of T exists, then a fortiori $T=pr\,N$, and consequently $T^*=prN$, from which it follows that

$$\left\|TT_0\right\|=\left\|NT_0\right\|=\left\|N^*T_0\right\|\geqq\left\|PN^*T_0\right\|=\left\|TT_0\right\|$$

for all $T\in\mathfrak{H}$. The inequality

$$\left\|TT_0\right\|\geqq\left\|T^*T_0\right\|\ \text{(for all } T_0\in\mathfrak{H}\text{)} \tag{1.4.1}$$

is, therefore, a necessary condition that T has a normal extension. However, it is easy to construct examples of operator T which does not satisfy this condition.
Other less simple necessary conditions are obtained in the following manner. Suppose $\left\{g_i\right\}(i=0,1,\ldots)$ is a sequence of elements in $\mathfrak{H}$ almost all of which (that is with perhaps the exception of a finite number of them) are equal to the element 0 in $\mathfrak{H}$. We then have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(T^i g_j, T^j g_i\right)=\sum_i\sum_j (N^i g_j, N^j g_i)=\sum_i\sum_j\left(N^{*j}N^i g_j g_i\right)$$

$$=\sum_i\sum_j\left(N^i N^{*j} g_j g_i\right)=\sum_i\sum_j\left(N^{*j} g_j, N^{*i} g_i\right)=\left\|\sum_i N^{*i} g_i\right\|^2 \geqq 0,$$

$$\sum_{i=o}^{\infty}\sum_{j=o}^{\infty}\left(T^{i+1} g_j, T^{j+1} g_i\right)=\left\|\sum_i \left(N^*\right)^{i+1} g_i\right\|^2 \leqq \left\|N^*\right\|^2 \sum_i N^* g_i\|^2$$

from which we see that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(T^i g_j, T^j g_i) \geqq 0 \tag{1.3.8}$$

and

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(T^{i+1} g_j, T^{j+1} g_i\right) \leqq C^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(T^i g_j, T^j, g_i\right) \tag{1.3.9}$$

with constant $C > 0$. These two inequalities are therefore necessary conditions that T has bounded normal extension. However, these conditions are also sufficient. Namely, the following theorem holds.
Theorem 1.4.1:[15] Every bounded linear operator in the Hilbert space $\mathfrak{H}$ which satisfies conditions (16) and (17) has a bounded normal extension N in an extension space H. One can even require that H be minimal in the sense that it is spanned by the elements of the form $N^{*k}T_0$ where $T_0 \in \mathfrak{H}$ and $k = 0,1,\ldots$; in this case, the structure $\{H, N, \mathfrak{H}\}$ is determined to within an isomorphism.
For the present, we shall concern ourselves with a remark connecting the problems on extensions.

Principal theorem
The following three propositions[15] are equivalent for any two bounded linear operators, T in $\mathfrak{H}$ and T_0 in $H\left(\supseteqq\mathfrak{H}\right)$:
a) $T \subseteqq T_0$;
b) $T = prT$ and $T^*T_0 = prT^*T_0$
c) $T^{*i}T_0^k = ptT^{*i}T_0^k$ for $i,k = 0,1,\ldots$

Proof: a) → c) because

$$\left(T^* T_0^k T_0, g\right)=\left(T^k T_0, T_0^i g\right)=\left(T^k g, T_0^i g\right)=\left(T^{*i} T_0^k T_0, g\right)=\left(P T_0^{*i} T_0^k T_0, g\right)$$

For $T, g \in \mathfrak{H}$. c) →b) is obvious. b) →a) is proved as follows: For $T' \in \mathfrak{H}$ we have on the other that

$$\|TT_0\|^2=\left(T^* T T_0, T_0\right)=\left(P T^* T T_0, T_0\right)=\left(T T_0, T T_0\right)=\|T T_0\|^2$$

because

$$T^*T = prT^*T,$$

and, on the other hand, that

$$\|TT_0\| = \|PTT_0\|$$

because

$$T = prT$$

Hence, we have

$$\|PTT_0\| = \|TT_0\|$$

Which is possible if and only if $TT_0 = PTT_0$, that is, if $TT_0 = TT_0$; therefore, $T \supseteqq T_0$

Let us first introduce some concepts of an algebraic nature. By a semi-group we shall understand a system Γ of elements (which we shall denote by Greek letters) in which two operations are defined: An associative "semi-group operation" $(\xi,\eta) \rightarrow \xi\eta$ and a "* operation," $\xi \rightarrow \xi^*$, which satisfies the following rules of computation:

$$\xi^{**} = \xi, \left(\xi r_i\right)^* = r_i^* \xi^*$$

We shall assume further that there is a unit element ε in Γ such that

$\varepsilon\xi = \xi\varepsilon = \xi$ for all $\xi \in \Gamma$, and $\varepsilon^* = \varepsilon$.

Any group can be considered as a semi-group if we define the $*$operation in it as the inverse: $\xi^* = \xi^{-1}$. In the sequel, when we speak of a group Γ, we shall assume that it is provided with this $*$-semi-group structure. By a representation of the $*$-semi-group Γ in a Hilbert space, H we shall understand a family $\left\{D_\xi\right\}_{\xi\in\Gamma}$ of bounded linear operators in H such that element whose images under the operators U_ξ span the space $\mathfrak{H}$; under these conditions, the structure $\left\{\mathfrak{H}, U_\xi, f_0\right\}$ is determined to within an isomorphism.

Proof of Theorem 1.4.1[16] (On Normal Extensions) Using the idea of the Principal Theorem

Now let Γ be the following $*$-semi-group: Its elements are the ordered pairs $\pi = \left\{i, j\right\}$ of non-negative integers; the semi-group operation is defined in it by

$$\pi + \pi' = \left\{i, j\right\} + \left\{i', j'\right\} = \left\{i + i', j + j'\right\}$$

and the $*$ operation by

$$\pi^* = \left\{i, j\right\}^* = \left\{j, i\right\};$$

then the "unit" element is

$$\varepsilon = \left\{0, 0\right\}$$

Let $\left\{D_\pi\right\}$ be a representation of Γ. Since the semi-group operation in Γ is commutative, the operator D_π is normal and pairwise doubly permutable. If we set $\eta = \{0,1\}$, every $\pi = \{i, j\}$ can be written in the form

$$\pi = i\eta^* + j\eta,$$

and consequently, we have

$$D_\pi = N^{*i} N^j$$

where

$$N = D_\eta$$

Let T be a bounded linear operator in the space $\mathfrak{H}$ which satisfies the conditions of theorem 2.3 on contraction

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(T^i g_j, T^j g_i\right) \geqq 0 \qquad (1.3.10)$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(T^{i+1} g_j, T^{j+1} g_i\right) \leqq C^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(T^i g_j, T^j g_i\right) \quad (C > 0) \qquad (1.3.11)$$

We[17] shall show that the operator

$$T_{\{i,j\}} = T^{*i}T^{j}$$

Considered as an operator defined on the *-semi-group Γ, satisfies the conditions of the principal theorem.

It is, first of all, obvious that $T_{\varepsilon} = I, T_{\pi^*} = T_{\pi}^{*}$. To prove that T_{π} is a function of positive type, we choose any family $\{g_{\pi}\}$ of elements in $\mathfrak{H}$, almost all of which are equal to 0, and consider the sum

$$s = \sum_{\pi}\sum_{\pi'}\left(T_{\pi^*+\pi'}g_{\pi'}, g_{\pi}\right)$$

where $\pi = \{i, j\}$ and $\pi' = \{i', j'\}$ run through the elements of Γ. It is easy to see that

$$s = \sum_{\pi}\sum_{\pi'}\left(\left(T^{*}\right)^{i'+j}T^{i+j'}g_{\pi'}, g_{\pi}\right) = \sum_{i}\sum_{i'}\left(T^{i}h_{i'}, T^{i'}h_{i}\right)$$

where

$$h = \sum_{j}T^{j}g_{\{i,j\}}$$

in virtue of (1.3.10), it follows from this that $s \geqq 0$ and therefore T_{π} is a function of positive type.

Repeating inequality (1.3.11) $(i_0 + j_0)$ times we obtain, in an analogous manner, that for fixed $\pi_0 = \{i_0 + j_0\}$, we have

$$\sum_{\pi}\sum_{\pi'}\left(T_{\pi^*+\pi_0^*+\pi_0+\pi'}g_{\pi'}, g_{\pi}\right) = \sum_{i}\sum_{j'}\left(T^{i_0+j_0+i}h_{i'}, T^{i_0+j_0'+i'}h_{i}\right)$$

$$\leqq C^{2(i_o+j_0)}\sum_{i}\sum_{i'}\left(T^{i}h_{i'}, T^{i'}h_{i}\right) = C^{2(i_0+j_0)}\sum_{\pi}\sum_{\pi'}\left(T_{\pi^*+\pi'}g_{\pi'}, g_{\pi}\right).$$

Conditions (a)-(c) of the principal theorem are therefore satisfied and then applying this theorem, it follows that in an extension space H, there exists a bounded normal operator N (with $\|N\| \leqq C$) such that

$$T^{*i}T^{i} = prN^{*i}N^{j} \quad (i, j = 0,1,\ldots),$$

which is equivalent to $T \subseteqq N$. If H is minimal in the sense that it is spanned by the elements $N^{*i}N^{j}T\left(T \in \mathfrak{H}\right)$, it is also spanned by the elements $N^{*i}T$, in as much as $N^{i}T \in \mathfrak{H}$ for $T \in \mathfrak{H}$. This completes the proof of theorem 1.4.1.

Proof of the Extension space of Theorem 1.4.2:[18] We denote the set of all the families $v = \{v_{\xi}\}_{\xi\in\Gamma}$ of elements v_{ξ} in $\mathfrak{H}$ by $V; v$ can be considered also as a vector whose component with index ξ is v_{ξ} in symbols:

$$(v)_{\xi} = v_{\xi}$$

The addition of these vectors and their multiplication by scalars (that is, by real or complex numbers according to as $\mathfrak{H}$ is real or complex) are defined by the corresponding operations on components.

In V we shall consider in particular two linear manifolds, G and F. G consists of vectors almost all of whose components are equal to 0; these vectors will be denoted by the letter g. F consists of vectors $T = \{T_i\}$ for which there exists a vector $g = \{g_i\}$ such that

$$T_{\xi} = \sum_{n}T_{\xi^*\eta}g_{\eta}$$

for all $\xi \in \Gamma$; this relationship between T and g will denoted by

$$T = \hat{g}$$

We define a binary form $[T,T]$ in F in the following way. If $T=\hat{g},T^{'}=\hat{g}^{'}$
we let

$$\left[T,T^{'}\right]=\sum_{\xi}\left(T_{\xi},g_{\xi}\right)=\sum_{\xi}\sum_{\eta}\left(T_{\xi^{*}\eta}g_{\eta},g_{\xi}^{'}\right) \tag{1.3.12}$$

$$=\sum_{\xi}\sum_{\eta}\left(g_{\eta},T_{\eta^{*}\xi}g_{\xi}^{'}\right)=\sum_{\eta}\left(g_{\eta},T_{\eta}^{'}\right) \tag{1.3.13}$$

(Here we have made use of the fact that $T_{\xi^{*}\eta}^{*}=T_{\left(\xi^{*}\eta\right)^{*}}=T_{\eta^{*}\xi}$). It follows from (1.3.12) that this definition does not depend on the particular choice of g in the representation of T, and it follows from (1.3.13) that it does not depend on the particular choice of $g^{'}$ either; consequently, the form $\left[T,T^{'}\right]$ is determined uniquely by T and $T^{'}$. It is obviously linear in T and we have $\left[T^{'},T\right]=\left[T,T^{'}\right]$. It follows from condition (b) that

$$\left[T,T\right]=\sum_{\xi}\sum_{\eta}\left(T_{\xi^{*}\eta}g_{\eta},g_{\xi}\right)\geqq 0.$$

We still have to prove that the equality sign holds here only for $T=0$. But it follows from what has already been proved that the Schwarz inequality is valid for the form $\left[T,T^{'}\right]$:

$$\left|\left[T,T^{'}\right]\right|^{2}\leqq[T,T][T^{'},T^{'}]$$

The equation $\left[T,T\right]=0$ for one T therefore implies that $\left[T,T^{'}\right]-0$ for all $T^{'}\in F$;

$$\left|\left[T,T^{'}\right]\right|^{2}\leqq[T,T][T,T^{'}]$$

The equation $\left[T,T\right]=0$ for one f therefore implies that $\left[T,T^{'}\right]=0$ for all $T\in F$; but it follows easily from (18) that this is possible only if $T=0$. Hence, the form $[T,T^{'}]$ possesses all the properties of a scalar product; therefore, if the scalar product in F is defined by

$$\left(T,T^{'}\right)=[T,T^{'}],$$

F becomes a Hilbert space, which in general is not complete. Let H be the completion of F. The original space $\mathfrak{H}$ can be embedded as a subspace in H, and even in F; this can be done by identifying the element T in $\mathfrak{H}$ with the element

$$T_{T}=\left\{T_{\xi^{*}}T\right\}$$

in F (note that $T_{T}=\hat{g}$ with $\left(g\right)_{\xi}=T$ and $\left(g\right)_{\xi}=0$ for $\xi\neq\varepsilon$). This identification is justified because we clearly have

$$T_{cT}=cT_{T},T_{T+T^{'}}=T_{T}+T_{T^{'}},\left(T_{T},T_{T^{'}}\right)=\left(T,T^{'}\right)$$

Let us now calculate the orthogonal projection PT of an element $T\in F$ onto subspace $\mathfrak{H}$! We should have for all $h\in\mathfrak{H}$,

$$\left(PT,h\right)=(T,h),$$

the definition of the scalar product in F yields

$$\left(PT,h\right)=\left(T,T_{h}\right)=\left(\left(T\right)_{\varepsilon},h\right);$$

since Pf and $(T)_{\varepsilon}$ are in $\mathfrak{H}$, this equation is possible for all $h \in \mathfrak{H}$ only if

$$PT = (T), \tag{1.3.14}$$

3. The representation $\{D_{\xi}\}$. Suppose $T = \hat{g}$, that is,

$$T_{\xi} = \sum_{\eta} T_{\xi^{*}\eta} g_{\eta}$$

We then have

$$T_{a^{*}\xi} = \Sigma_{\eta} T_{\xi^{*}a\eta} g_{\eta} \Sigma_{\xi} T_{\xi^{*}\xi} g_{\xi}^{a}$$

For arbitrary $a \in \Gamma$, where

$$g_{\xi}^{a} = \sum_{a\eta} g_{\eta} \tag{1.3.15}$$

(if there are no η such that $a\eta = \xi$, then the sum in the second member of (1.3.15) is defined to be equal to 0). It is clear that, for given $a, g_{\xi}^{a} = 0$ for almost all the ξ, and therefore

$$\{g_{\xi}^{a}\} \in G, \{T_{a^{*}\xi}\} \in F$$

Consequently,

$$D_{a}\{T_{\xi}\} = \{T_{a^{*}\xi}\}$$

is a operator, which is obviously linear of F into F. We have

$$D_{\xi}\{T_{\xi}\} = \{T_{\varepsilon^{*}\xi}\} = \{T_{\xi}\} \tag{1.3.16}$$

$$D_{a}D_{\beta}\{T_{\xi}\} = D_{a}\{T_{\beta^{*}\xi}\} = \{T_{\beta^{*}a^{*}\xi}\} = \{T_{(a\beta)^{*}\xi}\} = D_{a\beta}\{T_{\xi}\} \tag{1.3.17}$$

and for $T = \hat{g}, T' = \hat{g}'$,

$$(D_{a}T, T') = \sum_{\xi}(T_{a^{*}\xi}, g'_{\xi}) = \sum_{\xi}\sum_{\eta}(T_{\xi^{*}a\eta} g_{\eta}, g'_{\xi})$$

$$= \sum_{\xi}\sum_{\eta}(g_{\eta}, T_{\eta^{*}a^{*}\xi} g'_{\xi}) = \sum_{\eta}(g_{\eta}, T'_{a\eta}) = (T, D_{a^{*}}T') \tag{1.3.18}$$

Finally, it follows from (1.3.18), (1.3.17), and condition (c) of the principal theorem that

$$(D_{a}T, D_{a}T) = (D_{a^{*}}D_{a}T, T) = (D_{a^{*}a}T, T) = \sum_{\xi}\sum_{\eta}(T_{\xi^{*}a^{*}a\eta} g_{\eta}, g_{\xi})$$

$$\leqq C_{a}^{2}\sum_{\xi}\sum_{\eta}(T_{\xi^{*}\eta} g_{\eta}, g_{\xi}) = C_{a}^{2}(T, T)$$

Hence, D_{a} is a bounded linear operator in $F, \|D_{a}\| \leqq C_{a}$, and consequently, it can be extended by a continuity to H. It follows from (1.3.17) to (1.3.18) that $\{D_{\xi}\}$ thus extended will be a representation of Γ in H.

Consider in particular an element T which belongs to $\mathfrak{H}, T = T$. We then have

$$D_{a}T = D_{a}\{T_{\xi^{*}}T\} = \{T_{\xi^{*}a}T\} \tag{1.3.19}$$

and hence by (1.3.14)

$$PD_aT = \left(D_aT\right)_\varepsilon = T_aT$$

This proves that

$$T_a = prD_a$$

It also follows from (1.3.19) that, for $T = \hat{g} \in F$

$$\left(T\right)_\xi = \sum_\eta T_{\xi^*\eta} g_\eta = \sum_\eta \left(D_\eta g_\eta\right)_\xi = \left(\sum_\eta D_\eta g_\eta\right)_\xi,$$

whence,

$$T = \sum_\eta D_\eta g_\eta$$

This means that F consists of finite sums of elements of the form $D_\eta g$ where $g \in \mathfrak{H}, \eta \in \Gamma;$ then these elements span the space, H and therefore, the extension space H is minimal. The representation $\left\{D_\xi\right\}$ of Γ which we have just constructed also satisfies proposition 2 and 3 of the principal theorem. This follows from the equation

$$\left(D_aT, T^{'}\right) = \sum_\xi \sum_\eta \left(T_{\xi^* a\eta} g_\eta, g_\xi^{'}\right)$$

(See [1.3.8]), valid for arbitrary $T = \hat{g}, f^{'} = \hat{g}^{'} \left(\in F\right)$, and from the obvious fact that if the relation

$$\left(D_aT, T^{'}\right) = \left(D_\beta T, T^{'}\right) + \left(D_\gamma T, T^{'}\right),$$

or the relation

$$\left(D_{a_n}T, T^{'}\right) \to \left(D_aT, T^{'}\right) \quad (n \to \infty)$$

is satisfied for $T, T' \in F$ and if, moreover, in the second case

$$\overline{\lim} \| D_{a_n} \| < \infty$$

the same relation is satisfied for all the elements T, T' in H

4. Isomorphism

It remains to investigate the problem: To what extent is the structure $\left\{H, D_\xi, \mathfrak{H}\right\}$ determined? To this end, let us consider any two representations of $\Gamma, \left\{D_\xi^{'}\right\}$ in H' and $\left\{D_\xi^{"}\right\}$ in $H"$, where $H^{'}$ and $H"$ are two extension spaces of $\mathfrak{H}$, and let us assume that

$$prD_\xi^{'} = T_\xi, \quad prD_\xi^{"} = T_\xi$$

Furthermore,[19] we shall assume that each of these extension spaces is minimal, i.e., that H' is spanned by the elements $D_\xi^{'} g$ and $H^{"}$ by the elements of $D_\xi^{"} g$, where $g \in \mathfrak{H}$ and $\xi \in \Gamma$.

Let

$$T_1^{'} = \sum_\xi D_\xi^{'} g_{1\xi}, \; T_2^{'} = \sum_\xi D_\xi^{'} g_{2\xi}$$

be two elements in $H^{'}$ (with $\left\{g_{1\xi}\right\}, \left\{g_{2\xi}\right\} \in G$) and let

$$T_1^{"} = \sum_\xi D_\xi^{"} g_{1\xi}, \; T_2^{"} = \sum_\xi D_\xi^{"} g_{2\xi}$$

Be elements in $H^{"}$. We have

$$\left(T_1^{'},T_2^{'}\right)=\sum_\xi\sum_\eta\left(D_\eta g_{1\eta},D_\xi^{'}g_{2\xi}\right)=\sum_\xi\sum_\eta\left(D_{\xi^*\eta}^{'}g_{1\eta},g_{2\xi}\right)=\sum_\xi\sum_\eta\left(T_{\xi^*\eta}g_{1\eta},g_{2\xi}\right)$$

and in an analogous manner

$$\left(T_1^{"},T_2^{"}\right)=\sum_\xi\sum_\eta\left(T_{\xi^*\eta}g_{1\eta},g_{2\xi}\right),$$

hence

$$\left(T_1^{'},T_2^{'}\right)=\left(T_1^{"},T_2^{"}\right)$$

Consequently, if we assign the elements

$$T'=\sum_\xi D_\xi'\ g_\xi,\ T^{"}=\sum_\xi D_\xi^{"}g_\xi \tag{1.3.20}$$

To the same, $\left\{g_\xi\right\}\in G$ this correspondence $T^{'}\leftrightarrow T"$ will be linear and isometric and it can then be extended by continuity to a linear and isometric mapping of all the elements of $H^{'}$ onto $H^{"}$.
In particular, by setting $g_\varepsilon=g$ and $g_\xi=0$ for, $\xi\neq\varepsilon$, we see that each element g of the common subspace $\mathfrak{H}$ corresponds to itself. For all $a\in\Gamma$, we have

$$D_a^{'}\sum_\xi D_\xi^{'}g_\xi=\sum_\xi D_{a\xi}^{'}g_\xi=\sum_\xi D_\xi^{'}g_\xi^a\leftrightarrow\sum_\xi D_\xi^{"}g_\xi^a=\sum_\xi D_{a\xi}^{"}g_\xi=D_a^{"}\sum_\xi D_\xi^{"}g_\xi$$

(See 21); hence, $T\leftrightarrow T^{"}$ implies that $D_a^{'}T^{'}\leftrightarrow D_a^{"}T^{"}$ for all $T^{'},T^{"}$ in the form (26), and then, in virtue of the community of the operator $D_a^{'},D_a^{"}$, for all $T^{'}\in H^{'}$ and $T^{"}\in H^{"}$.
Therefore, the structures $\left\{H^{'},D_\xi^{'},\mathfrak{H}\right\}$ and $\left\{H^{"},D_\xi^{"},\mathfrak{H}\right\}$ are isometric. This completes the proof of this theorem.

CONTRACTIONS IN HILBERT SPACE

1. Whereas the projections of bounded self-adjoint operators are also self-adjoint, the projections of unitary operators are already of a more general type. In order that $T=prU$, with U unitary, it is necessary that

$$\|TT_0\|=\|PUT_0\|\leqq\|UT_0\|=\|T_0\|$$

for all $T_0\in\mathfrak{H}$, that is, $\|T\|\leqq1,$ and hence the operator T must be a contraction. However, this condition is not only necessary but also sufficient.
Theorem 2.1:[20,21] Every contraction T in the Hilbert space $\mathfrak{H}$ can be represented in an extension space H as the projection of a unitary operator U onto $\mathfrak{H}$. The theorem and the following simple construction of U, are due to Halmos. As in sec. 3 let us consider the product space $H=\mathfrak{H}\times\mathfrak{H}$ and the following operator of H;

$$U=\begin{pmatrix}T & S\\ -Z & T^*\end{pmatrix}\text{ where }S=\left(I-TT^*\right)^{\frac12},Z=\left(I-T^*T\right)^{\frac12} \tag{2.1}$$

The relation $T=prU$ is obvious. We shall show that U is unitary, or what amounts to the same thing, that U^*U and UU^* are equal to the identity operator I in H. Since S and Z are self-adjoint, we have

$$U^*U=\begin{pmatrix}T^* & -Z\\ S & T\end{pmatrix}\begin{pmatrix}T & S\\ -Z & T^*\end{pmatrix}=\begin{pmatrix}T^*T+Z^2 & T^*S-ZT^*\\ ST-TZ & S^2+TT^*\end{pmatrix}$$

$$UU^*=\begin{pmatrix}T & S\\ -Z & T^*\end{pmatrix}\begin{pmatrix}T^* & -Z\\ S & T\end{pmatrix}=\begin{pmatrix}TT^*+S^2 & -TZ+ST\\ -ZT^*+T^*S & Z^2+T^*T\end{pmatrix}$$

Since $Z^2 = I - T^*T, S^2 = I - TT^*$, the diagonal elements of the product matrices are all equal to I. It remains to show that the other elements are equal to 0, i.e., that

$$ST = TZ \tag{2.2}$$

(The equation $T^*S = ZT^*$ follows from this by passing over to the adjoints of both members of [2.2]). However, we have

$$S^2T = \left(I - TT^*\right)T = T - TT^*T = T\left(I - T^*T\right) = TZ^2,$$

from which it follows by complete induction that

$$S^{2n}T = TZ^{2n} \text{ for } n = 0,1,2,\dots$$

Then, we also have

$$p\left(S^2\right)T = Tp\left(T^2\right)$$

for every polynomial $p(\lambda)$. Since S and Z are the positive square roots of S^2 and Z^2, respectively, there exists a sequence of polynomials $p_n(\lambda)$ such that

$$p_n\left(S^2\right) \to S, p_n\left(Z^2\right) \to Z.$$

Now (12) follows from the equation

$$p_n\left(S^2\right)T = Tp_n\left(Z^2\right)$$

by passing to the limit as $n \to \infty$. This completes the proof of the theorem

2. The relationship between operators S in an extension space H of the space $\mathfrak{H}$ and their projections $T = prS$ onto $\mathfrak{H}$ is not multiplicative in general, that is the equations $T_1 = prS_1, T_2 = prS_2$ do not in general imply $T_1T_2 = prS_1S_2$. For example, if we consider the operator U constructed according to formula (2.1), we have $prU^2 = T^2 - SZ$, which in general is not equal to T^2.

The question arises: Is it possible to find, in a suitable extension space, a unitary operator U such that the powers of the contraction T (which are themselves contractions) are at the same time equal to the projections onto $\mathfrak{H}$ of the corresponding powers of U?

If we are dealing with only a finite number of powers,

$$T, T^2, \supset, T^k$$

then the problem can be solved in the affirmative in a rather simple manner suitably generalizing the immediately preceding construction. Let us consider the product space $H = \mathfrak{H} \times \ldots \times \mathfrak{H}$, with $k+1$ factors, whose elements are ordered $(k+1)$-tuples $\left\{T_1, \ldots, T_{k+1}\right\}$ of elements in $\mathfrak{H}$ and in which the vector operations and metric are defined in the usual way:

$$c\left\{T_1, \ldots, T_{k+1}\right\} = \left\{cT_1, \ldots, cT_{k+1}\right\},$$

$$\left\{T_1, \ldots, T_{k+1}\right\} + \left\{g_1, \ldots, g_{k+1}\right\} = \left\{T_1 + g_1, \ldots, T_{k+1} + g_{k+1}\right\}$$

$$\left(\left\{T_1, \ldots, T_{k+1}\right\}, \left\{g_1, \ldots, g_{k+1}\right\}\right) = \left(T_1, g_1\right) + \ldots + \left(T_{k+1}, g_{k+1}\right)$$

We embed $\mathfrak{H}$ in H as subspace of the latter by identifying the element T in $\mathfrak{H}$ with the element $\left\{T, 0, \ldots, 0\right\}$ in H. The bounded linear operator T in H will be represented by matrices $\left(T_{ij}\right)$ with $k+1$

rows and $k+1$ columns, all of whose elements T_{ij} are bounded linear operators in $\mathfrak{H}$. We have $T = prT$ if and only if $T_{11} = T$.

Let us now consider the following operators in H:

$$U = \left.\begin{pmatrix} T & S & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -1 \\ -Z & T^{*} & 0 & \dots & 0 & 0 \end{pmatrix}\right\} k+1 \, rows\, and\, columns$$

Where S and Z have the same meaning as in the foregoing construction. The operator U is unitary. This is proved in the same way as above, by a direct calculation of the matrices $U^{*}U, UU^{*}$. To prove the relations

$$T^{n} = prU(n = 1, 2, \dots, k).$$

We must calculate the element in the matrix U^{n} having indices 1, 2, and then note that the latter is equal to T^{n} for $n = 1, \dots, k$. We shall even prove more, namely, that the first row in the matrix $U^{n}\left(n = 1, \dots, k\right)$ is the following:

$$\left(T^{n}, T^{n-1}S, -T^{n-2}S^{2}, T^{n-3}S^{3}, \dots, \left(-1\right)^{n-1} S^{n}, \overbrace{0 \dots 0}^{k-n}\right).$$

This proposition is obvious for $n = 1$, and we prove it true for $n+1$, assuming it true for $n\left(n \leqq k-1\right)$ by calculating the matrix U^{n+1} as the matrix product $U^{n}.U$. We have thus proved the following theorem.

Theorem 2.2:[22] If T is a contraction in the Hilbert space $\mathfrak{H}$, then there exists a unitary operator U in an extension space H such that

$$T^{n} = prU^{n} \quad n = 0, 1, \dots, k$$

(the case $n = 0$ is trivial), for every given natural number k. The product space $\mathfrak{H} \times \dots \times \mathfrak{H}$ with $k+1$ factors can be written for H.

3. It is important in the above construction that k is a finite number; however, the theorem is also true for $k = \infty$.

Proof: Suppose Γ is the additive group of all integers n. Every representation of Γ is then of the form $\left\{U^{n}\right\}$ where U is a unitary operator. Suppose T is a contraction in $\mathfrak{H}$. Set

$$T_{n} = \begin{cases} T^{n} & for\, n = 0, 1, \dots, \\ T^{*|n|} & for\, n = 0, 1, \dots,; \end{cases}$$

hence $T_{0} = I$ and $T_{-n} = T_{n}^{*}$. We shall show that T_{n}, considered as function defined on Γ, is of positive type, that is

$$\sum\nolimits_{m} \sum\nolimits_{n} \left(T_{n-m} g_{n}, g_{m}\right) \geqq 0 \tag{2.3}$$

for every sequence $\left\{g_{n}\right\}_{-\infty}^{\infty}$ of elements in $\mathfrak{H}$ almost all of which are equal to 0. We first consider the case of a complex space $\mathfrak{H}$. We set

$$T\left(r, \varphi\right) = \sum\nolimits_{-\infty}^{\infty} r^{n} e^{in\varphi} T_{n} \tag{2.4}$$

for $0 \leqq r < 1$ and $0 \leqq \varphi \leqq 2\pi$; in view of the fact that $\|T_n\| \leqq 1$, this series converges in norm. Setting $z = re^{i\varphi}$ we have

$$T(r,\varphi) = \left(\frac{1}{2}I + \sum_1^\infty z^n T^n\right) + \left(\frac{1}{2}I + \sum_1^\infty \bar{z}^n T^{*n}\right)$$

$$= 2\mathrm{Re}\left(\frac{1}{2}I + \sum_1^\infty z^n T^n\right) = \mathrm{Re}(I + zT)(I - zT)^{-1}$$

Hence, for $T \in \mathfrak{H}$ and $g = (I - zT)^{-1} T,$ we have in particular that

$$(T(r,\varphi)T,T) = \mathrm{Re}\left((I + zT)(I - zT)^{-1} T, T\right) = \mathrm{Re}\left((I + zT)g, (I - zT)g\right)$$

$$= \mathrm{Re}\left[(g,g) + z(Tg,g) - \bar{z}(g,Tg) - z\bar{z}(Tg,Tg)\right] = \|g\|^2 - |z|^2\|Tg\|^2 \geqq 0 \quad (33)$$

since $|z| < 1, \|T\| \leqq 1.$ Since this result holds for all $T \in \mathfrak{H}$, we have in particular that

$$p(r,\varphi) = \left(T(r,\varphi)T(\varphi), T(\varphi)\right) \geqq 0 \tag{2.5}$$

with

$$T(\varphi) = \sum_{n=-\infty}^{\infty} e^{-in\varphi} g_n$$

where $\{g_n\}$ is the sequence of elements in $\mathfrak{H}$ considered in inequality (2.3). If in (2.6) we replace $T(r,\varphi)$ and $T(\varphi)$ by their series expansions, we obtain that

$$p(r,\varphi) = \sum_{k,m,n} r^{|k|} e^{i(k+m-n)\varphi}\left(T_k g_n, g_m\right) =$$

$$\sum_I e^{iI\varphi} \sum_{m,n} r^{|I+n-m|}\left(T_{I+n-m} g_{n,} g_m\right) \geqq 0$$

whence,

$$\sum_{m,n} r^{|n-m|}\left(T_{n-m} g_n, g_m\right) = \frac{1}{2\pi}\int_0^{2\pi} p(r,\varphi)\,d\varphi \geqq 0$$

and, on the other hand, we have

$$\left\|\bar{T}\{g,h\}\right\|^2 = \left\|\{Tg,Th\}\right\|^2 = \|Tg\|^2 + \|Th\|^2 \leqq \|g\|^2 + \|h\|^2 = \left\|\{g,h\}\right\|^2$$

It is also easily seen that

$$\bar{T}^*\{g,h\} = \{T^*g, T^*h\}.$$

It follows that

$$\bar{T}^{"}\{g,h\} = \{T^{"}g, T^{"}h\} \text{ and } \bar{T}^{*"}\{g,h\} = \{T^{*"}g, T^{*"}h\}$$

for $n = 0,1,\ldots$ and hence

$$\bar{T}_n\{g,h\} = \{T_n g, T_n h\}$$

For $n = 0, \pm 1, \pm 2, \ldots$

But, since quality (2.3) has already been proved for complex spaces, we shall have

$$\sum_m \sum_n \left(\bar{T}_{n-m}\varphi_n, \varphi_m\right) \geqq 0 \tag{2.7}$$

For $\varphi_n = \{g_n, h_n\}$ (where $g_n = 0$ and $h_n = 0$ for almost all n). When $h_n = 0$ for all n, we have

$$\left(\bar{T}_{n-m}\varphi_n, \varphi_m\right) = \left(T_{n-m} g_n, g_m\right),$$

and hence inequality (2.7) then reduces to inequality (2.3), which completes the proof of (2.3) also in the case of a real space $\mathfrak{H}$.
We can then apply the principal theorem, which yields theorem 1.3.
Theorem 2.3:[23] If T is a contraction in the Hilbert space $\mathfrak{H}$, then there exists a unitary operator U of an extension space H such that the relation

$$T^n = prU^n$$

is valid for $n = 0,1,2,\ldots$ Furthermore, one can require that the space H be minimal in the sense that it is spanned by the elements of the form $U^n T$ where $T \in \mathfrak{H}$ and $n = 0, \pm 1, \pm 2, \ldots,$ in this case, the structure $\{H, U, \mathfrak{H}\}$ is determined to within an isomorphism.
An analogous theorem is true for semi-groups and one-parameter semi-groups of contractions, that is for families $\{T_1\}$ of contractions (where $0 \leqq t < \infty$) or $-\infty \leqq t < \infty$, according to the case at hand) such that

$$T_0 = I, T_{t_1} T_{t_2} = T_{t_1 + t_2},$$

And for which one assumes further that T_t depends strongly or weakly continuously on t; weak continuity means that (T, f, g) is a continuous numerical-valued function of t for every pair T, g of elements $\mathfrak{H}$.
Proof: Now, let Γ be the additive group of all real numbers t. Then, the representations of Γ are one-parameter groups $\{U_t\}$ of a unitary operator.
Let $\{T_t\}_{t \geqq 0}$ be the one-parameter semi-group of contractions considered in theorem. We set

$$T_t = T_{-t}^*$$

for $t < 0;$ then T_t will be weakly continuous function of $t, -\infty < t < \infty,$ and we shall have

$$T_0 = I \text{ and } T_{-t} = T_t^* \text{ for } -\infty < t < \infty.$$

We[23] shall show that T_t, considered as a function on Γ, is of positive type, that is,

$$\sum_s \sum_t \left(T_{t-s} h_t, h_s\right) \geqq 0 \tag{2.8}$$

for every family $\{h_t\}$ of elements in $\mathfrak{H}$ such that $h_t = 0$ for almost all values of t. Suppose $t_1, t_2, \ldots, t_r$ are those values of t for which $h_t \neq 0$. We assign to each $t_n \left(n = 1, \ldots, r\right)$ a sequence of rational numbers $t_{nv}\left(v = 1, 2, \ldots,\right)$ which converges to t in such a manner that the numbers $t_{nv}(n = 1, 2, \ldots, r)$ are distinct for every fixed index v. Since T_t is a weakly continuous function of $t,$ setting

$$f_n = h_{t_n} \quad \left(n = 1, 2, \ldots, r\right),$$

We have

$$\sum_s \sum_t \left(T_{t-s} h_t, h_s\right) = \sum_{m=1}^r \sum_{n=1}^r \left(T_{t_n - t_m} f_n, f_m\right)$$

$$= \lim_{v \to \infty} \sum_{m=1}^r \sum_{n=1}^r \left(T_{t_{nv} - t_{mv}} f_n, f_m\right) \tag{2.9}$$

For every fixed v, the rational numbers $t_{nv}(n=1,2,\ldots,r)$ are commensurable, that is they can be written in the form

$$t_{nv}=\tau_{nv}d_v$$

with a $d_v>0$ and distinct integers τ_{nv}. Then, we have

$$T_{t_{nv}-t_{mv}}=T_{\left(r_{nv}-t_{mv}\right)dv}=\begin{cases}\left(T_{d_v}\right)^{\tau_{nv}-\tau_{mv}} & \text{when}\,\tau_{nv}\geqq\tau_{mv}\\ \left(T^*_{d_v}\right)^{\tau_{mv}-\tau_{nv}} & \text{when}\,\tau_{nv}\leqq\tau_{mv}\end{cases} \tag{2.10}$$

Where $T_n^{(v)}$ is defined in a manner analogous to (2.2), starting with the transformation $T^{(v)}=T_{d_v}$. Since the latter is a contraction, inequality (2.3) holds for it also; choosing the g_n in (2.3) in such a way that

$g_n=f_p$ when $n=\tau_{pv}$,

$g_n=0$ when n is not equal to any of the $\tau_{qv}\left(q=1,2,\ldots,r\right)$,

the first member of inequality (2.2) reduces to the second member of equation (2.10), and hence the latter is $\geqq 0$; and this is true for all fixed values of v. Inequality (2.8) follows, in virtue of (2.9).
Then, we can apply the principal theorem and obtain that

$$T_t=prU_t,$$

and that in the case where the extension space H in question is minimal, the structure $\{H,U_t,\mathfrak{H}\}$ is determined to within an isomorphism. In this case, U_t is also a weakly (and hence strongly) continuous function of t, and this in virtue of proposition (3) of the principal theorem and because of the fact that T_{t+t_n} is obviously a weakly continuous function of t for and arbitrary fixed value t_0 of t. This completes the proof of theorem 2.3.
Theorem 2.4:[23] If $\{T_t\}_{t\geqq 0}$ is a weakly continuous one-parameter semi-group of contractions in the Hilbert space $\mathfrak{H}$, then there exists a one-parameter group $\{U_t\}_{-\infty<t<\infty}$ of a unitary operator in an extension space H, such that

$$T_t=prU_t \text{ for } t\geqq 0$$

Furthermore, one can require that the space H be minimal in the sense that it is spanned by elements of the form U_tf, where $f\in\mathfrak{H}$ and $-\infty<t<\infty$; in this case, U_t is strongly continuous and the structure $\{H,U_t,\mathfrak{H}\}_{-\infty<t<\infty}$ is determined to within an isomorphism.
These two theorems can be generalized to discrete or continuous semi-groups with several generators. We shall formulate only the following generalization of theorem 2.3.
Proof: We now choose " to be the group of all the "vectors" $n=\left\{n^{(\rho)}\right\}_{\rho\in R}$ whose components are integers, almost all of which are equal to 0. If $\left\{T^{(\rho)}\right\}_{\rho\in R}$ is the given system of pairwise doubly permutable contractions, we set

$$T_n=\prod\nolimits_{\rho\in R}T^{(\varrho)}_{n(\rho)} \tag{2.11}$$

Where $T_n^{(\rho)}$ is defined in a manner analogous to (2.2). Since $n^{(\rho)}=0$ for almost all ρ, almost all the factors in the product (2.11) are equal to I; therefore, product has meaning even in the case where the set R in infinite. We note which is essential that since they $T^{(\rho)}$ are pairwise doubly permutable, the factors in (2.11) are all permutable. We obviously have $T_o=I,T_{-n}=T_n^*$, where o denotes the vector all of whose components equal zero.
It remains to prove that T_n, considered as a function on the group Γ, is of positive type, that is

$$\sum_m \sum_m \left(T_{n-m} g_n, g_m\right) \geqq 0 \tag{2.12}$$

for every family $\{g_n\}$ of elements in $\mathfrak{H}$ such that $g_n = 0$ for almost all $n \in \Gamma$. If one considers only those vectors n for which $g_n \neq 0$, there is a finite number of indices ρ, say $\rho_1, \rho_2, \ldots, \rho_r$, such that all the components of the vectors n whose indices are different from these are equal to 0. Since the factors with $n^{(\rho)} = 0$ in the product (2.11) can be omitted, it suffices to consider the sums of the type

$$\sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_r=-\infty}^{\infty} \left(T_{n_1-m_1}^{(1)} \ldots T_{n_r-m_r}^{(r)} g_{n_1,\ldots,n_r}, g_{m_1,\ldots,m_r}\right) \tag{2.13}$$

where we have set $T^{(i)}$ in place of $T^{(\rho_i)}$ for simplicity in writing. In the case of a complex space $\mathfrak{H}$ one can reason as follows. We set

$$T\left(r, \varphi_1, \ldots, \varphi_r\right) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_r=-\infty}^{\infty} r^{|n_1|+\ldots+|n_r|} e^{i(n_1\varphi_1+\ldots+n_r\varphi_r)} T_{n_1}^{(1)} \ldots T_{n_r}^{(r)}$$

$$= \prod_{i=1}^{r} T^{(i)}\left(r, \varphi_i\right),$$

for $0 \leqq r < 1$ and $0 \leqq \varphi_i \leqq 2\pi$, where the factors in the last member have a meaning analogous to (2.4). Since these factors are, according to (2.13), ≧0, and since they are pairwise permutable, their product is also $\geqq O$. Hence, we have in particular that

$$\left(T\left(r, \varphi_1, \ldots, \varphi_r\right) g\left(\varphi_1, \ldots, \varphi_r\right), g\left(\varphi_1, \ldots, \varphi_r\right)\right) \geqq 0$$

with

$$g\left(\varphi_1, \ldots, \varphi_r\right) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_r=-\infty}^{\infty} e^{-i(n_1\varphi_1+\ldots+n_r\varphi_r)} g_{n_1,\ldots,n_r}$$

Integrating with respect to each variable φ_i from $2\neq$, and then letting r tend to I, we have the result that the sum (2.13) is ≧0.

This proves inequality (2.12) for the case of complex space. The case of a real space can be reduced to that of a complex space in the same way this was done in the proof of 2.2.

The principal theorem can then be applied. To obtain theorem 2.3, it only remains to observe that every representation $\{U_n\}$ of the group Γ is of the form

$$U_n = \prod_{\rho \in R} \left[U^{(\rho)}\right]^{n(\rho)} \qquad \left(n = \left\{n^{(\rho)}\right\}\right)$$

where $\left\{U^{(\rho)}\right\}$ is a system of permutable unitary operator. This follows from the fact that n can be written in the form

$$n = \sum_{\rho \in R} n^{(\rho)} e_\rho$$

where e_ρ denotes the vector all of whose components equal zero except the component with index ρ, which s equal to 1; all one has to do is set

$$U^{(\rho)} = U_{e_\rho}$$

Theorem 2.5:[23] Suppose $\left\{T^{(\rho)}\right\}_{\rho \in R}$ is a system of pairwise doubly permutable contraction in the Hilbert space $\mathfrak{H}$. There exists in an extension space H, a system $\left\{U^{(\rho)}\right\}_{\rho \in R}$ of pairwise unitary transformations such that

$$\prod_{i=1}^{r} \left[T^{(\rho_i)}\right]^{n_i} = pr \prod_{i=1}^{r} \left[U^{(\rho_i)}\right]^{n_i}$$

for arbitrary $\rho_i \in R$ and integers n_i, provided the factor $\left[T^{(\rho_i)}\right]^{n_i}$ is replaced by $\left[T^{(\rho_i)^*}\right]^{-n_i}$ when $n_i < 0$. Moreover, one can require that the space H be minimal in the sense that it be spanned by the elements of the form $\prod_{i=1}^{r}\left[U^{(\rho_i)}\right]^{n_i} T$ where $T \in \mathfrak{H}$; in this case, the structure $\left\{H, U^{(\rho)}, \mathfrak{H}\right\}_{\rho\in R}$ is determined to within an isomorphism

Proof: Before this proof, we consider applications of the theorem as stated below

a. Invariant elements. If the element t is invariant with respect to T, then it is also invariant with respect to T^*

Proof of (a): We have $T = prU$, with U unitary from which it follows that $T^* = prU^* = prU^{-1}$. The equations $T = TT = PUT$, $\|UT\| = \|T\|$ imply that $UT = T$. Hence, we have $T = U^{-1}T = PU^{-1}T = T^*T$, which completes the proof of the theorem.

b. Ergodic theorems. For all $T \in \mathfrak{H}$ the limits

$$\lim_{\substack{n>m\geqq 0\\ n-m\to\infty}} \frac{1}{n-m}\sum_{k=m}^{n-1} T_{k\,T}$$

and

$$\lim_{\substack{v>\mu\geqq 0\\ v-\mu\to\infty}} \frac{1}{r-\mu}\int_{\mu}^{v} T_t T dt$$

exist in the sense of strong convergence of elements, where the integral is defined as the strong limit of sums of Riemann type

Proof: By theorems 4.3 and 4.4, we have $T^k = prU^k (k = 0,1,\ldots)$ and $T_t = prU_t \left(t \geqq 0\right)$ with U_t strongly continuous; hence, T_t is also strongly continuous. For $T \in \mathfrak{H}$, we have

$$\sum_{m}^{n-1} T^k T = P\sum_{m}^{n-1} U^k T$$

and

$$\int_{\mu}^{v} T_t T dt = P\int_{\mu}^{v} U_t T dt$$

respectively, and the propositions thus follow from the ergodic theorems of J. Von Neumann on unitary operators. Dunford's ergodic theorem on several permutable contractions can be reduced in an analogous manner, by theorem 2.5, to the particular case of the unitary operator, but this only under the additional condition that these contractions be doubly permutable

c. Theorems of VON NEUMANN and HEINZ. Suppose

$$u\left(z\right) = c_0 + c_1 z + \ldots + c_n z^n + \ldots$$

Is a power series in the complex variable z with

$$\left|c_0\right| + \left|c_1\right| + \ldots + \left|c_n\right| + \ldots < \infty$$

Set

$$u\left(T\right) = c_0 I + c_1 T + \ldots + c_n T^n + \ldots$$

If the function $u(z)$ satisfies one or the other of the inequalities

$$\left|u\left(z\right)\right| \leqq 1, \operatorname{Re} u\left(z\right) \geqq 0, \text{with } |z| \leqq 1,$$

then we have

$$\|u(T)\| \leqq 1, \quad Re\, u(T) \geqq 0$$

respectively.
Proof: It follows from the representation of powers: $T^k = prU^k (k = 0,1,\ldots)$ that

$$U(T) = pr\, u(U)$$

Let

$$U = \int_0^{2\pi} e^{\lambda i} dE_\lambda$$

be the spectral decomposition of the unitary operator U; we then have

$$\|u(T)T_0\|^2 = \|Pu(U)T_0\|^2 \leqq \|u(U)T_0\|^2 = \int_0^{2\pi} \left|u\left(e^{i\lambda}\right)\right|^2 d\left(E_\lambda T_0, T_0\right),$$

$$\mathrm{Re}\left(u(T)T_0, T_0\right) = \mathrm{Re}\left(Pu(U)T_0, T_0\right) = \mathrm{Re}\left(u(U)T_0, T_0\right) = \int_0^{2\pi} u\left(e^{i\lambda}\right) d\left(E_\lambda T_0, T_0\right)$$

for $T_0 \in \mathfrak{H}$. The above propositions follow in an obvious manner from formulae
d. Let

$$p(\theta) = \sum_k a_k e^{it} k^\theta$$

Be a trigonometric series with arbitrary real t_k and such that

$$\sum_k |a_k| < \infty \tag{2.14}$$

Set

$$p(T) = \sum a_k T_{t_k}$$

If the function $p(\theta)$ satisfies one or the other of the inequalities

$|p(\theta)| \leqq 1, Re\, p(q) \geqq 0$ for all real θ,

then

$$\|p(T)\| \leqq 1, \mathrm{Re}\, p(T) \geqq 0$$

respectively.
Proof: The proof proceeds exactly as for (c) but now using Theorem 4.4 and Stone's theorem in virtue of which there is a spectral decomposition of U_t of the form

$$U_t = \int_{-\infty}^{\infty} e^{it\lambda} dE_\lambda$$

Analogous theorems could be stated (under suitable hypotheses which assure convergence) for trigonometric integrals.

4. Isometric operators in Hilbert space $\mathfrak{H}$ (into a subspace of $\mathfrak{H}$) are particular cases of contractions. If the isometric operator T is represented as the projection of a unitary operator U, we have

$$\|T\| = \|TT_0\| = \|PUT_0\| \leqq \|UT_0\|$$

for all T; since, on the other hand, $\|UT_0\| = \|T\|$, we necessarily have $PUT_0 = UT_0$, and hence $TT_0 = UT_0$; that is, U is an extension of T.

It, therefore, follows from our theorems on contraction that every isometric transformation has a unitary extension and that for every weakly continuous one-parameter semi-group of isometric operator T_t, there exists a strongly continuous one-parameter group of unitary transformation U_t in an extension space such that $U_t \supseteq T_t$.

The last theorem was proved earlier by COOPER in an entirely different way.

APPLICATION OF THE ANALYTIC CONTRACTION AND EXTENSION TO ELASTICITY DEFORMATION AND STRESS

Deformation and stress

In this section, we discuss how the position of each particle may be specified at each instant and we introduce certain measures of the change of shape and size of infinitesimal elements of the material. These measures are known strains and they are used later in the derivation of the equations of elasticity. We also consider the nature of the forces acting on arbitrary portions of the body and this leads us into the concept of stress.

Motion, material, and spatial coordinates

We wish to discuss the mechanics of bodies composed of various materials. We idealize the concept of a body by supposing that it is composed of a set of particles such that, at each instant of time t, each particle of a set is assigned to a unique point of a closed region ℓ_t of three-dimensional Euclidean space and that each point of ℓ_t is occupied by just one particle. We call ℓ_t the configuration of the body at time t.

To describe the motion of the body, that is, to specify the position of each particle at each instant, we require some convenient method of labeling the particles. To do this, we select one particle configuration ℓ and call this the reference configuration. The set of coordinates (X_1, X_2, X_3), or position vector X, referred to fixed Cartesian axes of a point of ℓ uniquely determines a particle of the body and may be regarded as a label by which the particle can be identified for all time. We often refer to such a particle as the particle X. In choosing, ℓ we are not restricted to those configurations occupied by the body during its actual motion, although it is often convenient to take ℓ to be the configuration ℓ_0 occupied by the body at some instant which is taken as the origin of the time scale t. The motion of the body may now be described by specifying the position x of the particle X at time t in the form of an equation

$$x = \chi(X, t) \tag{3.1}$$

[Figure 1] or in component form,

$$x_1 = \chi_1(X_1, X_2, X_3, t), x_2 = \chi_2(X_1, X_2, X_3, t), x_3 = \chi_3(X_1, X_2, X_3, t) \tag{3.2}$$

And we assume that the functions χ_1, χ_2, and χ_3 are differentiable with respect to X_1, X_2, X_3 and t as many as required. Sometimes we wish to consider only two configurations of the body, an initial configuration and a final configuration. We refer to the mapping from the initial to the final configuration as a deformation of the body, which is either contraction or expansion. The motion of the body may be regarded as a one-parameter sequence of deformations.

We assume that the Jacobian

$$J = \det\left(\frac{\partial \chi_i}{\partial X_A}\right), \quad i, A = 1, 2, 3 \tag{3.3}$$

Exist at each point of ℓ_t, and that

$$J > 0 \tag{3.4}$$

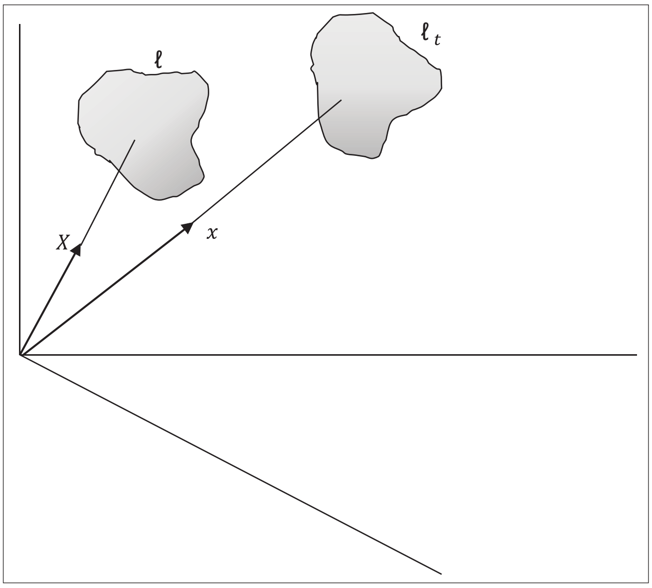

Figure 1: Cartesian coordinate of a typical particle

The physical significance of these assumptions is that the material of the body cannot penetrate itself and that material occupying a finite non-zero volume in ℓ cannot be compressed to a point or expanded to infinite volume during the motion. Mathematically (3.4) implies that (3.1) has the unique inverse

$$X = \chi^{-1}(x,t) \tag{3.5}$$

Now at the current time t the position of a typical particle P is given by its Cartesian coordinates (x_1, x_2, x_3), but as mentioned above, P continues to be identified by the coordinates (X_1, X_2, X_3) which denoted its position in ℓ. The coordinates (X_1, X_2, X_3) are known as material (or Lagrangian) coordinates since distinct sets of these coordinates refer to distinct material particles. The coordinates (x_1, x_2, x_3) are known as spatial (or Eulerian) coordinates since distinct sets refer to distinct points of the space. The values of x given by equation (3.1) for a fixed value of X are those points of space occupied by the particle X during the motion. Conversely, the values of X given by equation (3.5) for a fixed value of x identify the particles X passing through the point x during the motion.
From now on, when upper or lower case letters are used as suffixes, they are understood generally to range over 1, 2, and 3. Usually, upper case suffixes refer to material coordinates, lower case to spatial and repetition of any suffix refer to summation over the range. For example, we write x_i for $(x_1, x_2, x_3), X_A$ for (X_1, X_2, X_3) and $x_i x_i$ denotes $x_1^2 + x_2^2 + x_3^2$.
When a quantity is defined at each point of the body at each instant of time, we may express this quantity as a function of X_A and t or of x_i and t. If X_A and t are regarded as the independent variables, then the function is said to be a material description of the quantity; if x_i and t are used then the corresponding function is said to be a spatial description. One description is easily transformed into the other using (3.1) or (3.5). The material description spatial description $\psi(X,t)$ has a corresponding spatial description $\psi(x,t)$ related by

$$\psi\left(\chi^{-1}(x,t),t\right) = \psi(x,t) \tag{3.6}$$

or

$$\psi\left(\chi(X,t),t\right) = \psi(X,t) \tag{3.7}$$

To avoid the use of a cumbersome notation and the introduction of a large number of symbols, we usually omit explicit mention of the independent variables and also use a common symbol for a particular quantity and regard it as denoting sometimes a function of X_A and t and sometimes the associated function of x_i and t. The following convention for partial differentiation should avoid any confusion.

Let u be the common symbol used to represent a quantity with the material description ψ and spatial description ψ (these may be scalar-, vector-, or tensor-valued functions) as related by (3.6) and (3.7). We adopt the following notation for the various partial derivatives:

$$u_K = \frac{\partial \psi}{\partial X_K}(X,t), \quad \frac{Du}{Dt} = \frac{\partial \psi}{\partial t}(X,t) \tag{6.8}$$

$$u_i = \frac{\partial \psi}{\partial x_i}(x,t), \qquad \frac{\partial u}{\partial t} = \frac{\partial \psi}{\partial t}(x,t) \tag{6.9}$$

The material time derivative

Suppose that a certain quantity is defined over the body and we wish to know its time rate of the change as would be recorded at a given particle X during the motion. This means that we must calculate the partial derivative with respect to time of the material description ψ of the quantity keeping X fixed. In other words, we calculate $\partial \psi(X,t)/\partial t$. This quantity is known as a material time derivative. We may also calculate the material time derivative from the spatial description ψ . Using the chain rule of partial differentiation, we see from (3.7) that

$$\frac{\partial \psi}{\partial t}(X,t) = \frac{\partial \psi}{\partial t}(x,t) + \frac{\partial \chi_i}{\partial t}(X,t)\frac{\partial \psi}{\partial x_i}(x,t) \tag{3.10}$$

remembering, of course, that repeated suffixes imply summation over 1, 2, and 3. Consider now a given particle X_0. Its position in the space at time t is

$$x = \chi(X_0, t)$$

and so its velocity and acceleration are

$$\frac{d\chi}{dt}(X_0,t) \text{ and } \frac{d^2\chi}{dt^2}(X_0,t)$$

respectively. We therefore define the velocity field for the particles of the body to be material time derivative $\partial \chi(X,t)/\partial t$, and use the common symbol v to denote its material or spatial description

$$v = \frac{\partial \chi}{\partial t}(X,t) = \frac{Dx}{Dt} \tag{3.11}$$

Likewise, we define the acceleration field f to be the material time derivative of v

$$f = \frac{Dv}{Dt} \tag{3.12}$$

Moreover, in view of (6.10) the material time derivative of u has the equivalent forms

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + v_i u_i \tag{3.13}$$

In particular, the acceleration (3.12) may be written as

$$f = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + (v.\nabla)v \tag{3.14}$$

where the operator ∇ is defined relative to the coordinates x_i , that is

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \tag{3.15}$$

In suffix notation (6.14) becomes

$$f_i = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j v_{i,j} \tag{3.16}$$

The deformation-gradient tensor

We have discussed how the motion of a body may be described. In this section, we analyze the deformation of infinitesimal elements of the body which results from this motion. Suppose that ℓ coincides with the initial configuration ℓ_0, and that two neighboring particles P and Q have positions X and $X + dX$ in ℓ. Then, at time t their positions in ℓ_t are x and $x + dx$. Where

$$x = \chi(X,t), \quad x + dx = \chi(X + dX, t) \tag{3.17}$$

and the components of the total differential dx are given in terms of the components of dX and the partial derivatives of χ by

$$dx_i = \frac{\partial \chi_i}{\partial X_A}(X,t)\, dX_A = x_{i,A} dX_A \tag{3.18}$$

The quantities $x_{i,A}$ are known as the deformation gradients. They are the components of a second-order tensor known as the deformation-gradient tensor, which we denote by F.

Strain tensors

Denoting the deformation gradients $x_{i,A}$ by $F_{i,A}$, equation (3.18) may be written

$$dx_i = F_{iA} dX_A \tag{3.19}$$

In view of our assumption (3.4), the tensor F is non-singular, and so permits the unique decompositions

$$F = RU, \quad F = VR \tag{3.20}$$

Where U and V are positive definite symmetric tensors and R is proper orthogonal. We note that a proper orthogonal tensor R has the properties

$$R^T R = RR^T = I, \quad \det R = 1 \tag{3.21}$$

Where R^T denotes the transpose of R, and I denotes the unit tensor. A positive definite tensor U has the property

$$x_i U_{ij} x_j > 0 \tag{3.22}$$

For all non-null vectors x. To see the physical significance of the decomposition (3.20), we first write (3.19) in the form

$$dx_i = R_{iK} U_{KL} dX_L \tag{3.23}$$

or equivalently,

$$dx_i = R_{iK} dy_K, \quad dy_K = U_{KL} dX_L \tag{3.24}$$

In other words, the deformation of line elements dX into dx caused by the motion, may split into two parts. Since U is a positive-definite symmetric tensor, there exists a set of axes, known as principal axes, referred to which U is diagonal; and the diagonal components are the positive principle values U_1, U_2, U_3 of U. Equation (3.24), referred to these axes, becomes

$$dy_1 = U_1 dX_1,\ dy_2 = U_2 dX_2,\ dy_3 = U_3 dX_3 \tag{3.25}$$

In the deformation represented by equations (3.25), the i^{th} component of each line element is increased or diminished in magnitude according to as $U_i > 1$ or $U_i < 1$. This part of the deformation
Therefore, amounts to a simple stretching or compression in three mutually perpendicular directions. (Of course, if $U_i = 1$ the corresponding component of the line element is unchanged). The values of U_i are known as the principal stretches. Equation (3.24) describes a rigid body rotation of the line elements dy to dx. Hence, the line elements dX may be thought of as being first translated from X to x, then stretched by the tensor U as described above, and finally rotated as a rigid body in a manner determined by R [Figure 2]. The decomposition (3.20) may be interpreted in a similar way, although it should be noted that in this case, the rotation comes before the stretching. The tensors U and V are known as the right and left stretching tensors, respectively.
Although the decomposition (3.20) provide useful measures of the local stretching of an element of the body as distinct from its rigid body rotation, the calculation of the tensors U and V for any but the simplest deformations can be tedious. For this reason, we define two more convenient measures of the stretching part of the deformation. We define the right and left Cauchy–Green strain tensors

$$C = F^T F, \quad B = FF^T \tag{3.26}$$

respectively. Clearly C and B are symmetric second-order tensors. The tensor C is easily related to U since using (3.20) and (3.21)

$$C = U^T R^T RU = U^T U = U^2 \tag{3.27}$$

Similarly, we can show that

$$B = V^2 \tag{3.28}$$

As can be seen from the definitions (3.26), when F has been found, the tensors B and C are easily calculated by matrix multiplication; and in principle, U and V can be determined as the unique positive-definite square roots of C and B are diagonal. In such cases, U and V can be found easily.
Example 3.1:
Find the tensors F, C, B, U, V and R for the deformation

$$x_1 = X_1, x_2 = X_2 - \alpha X_3, x_3 = X_3 + \alpha X_2 \tag{3.29}$$

where $\alpha(>0)$ is a constant and interpret the deformation as a sequence of stretches and a rotation.
For this deformation

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{pmatrix}, \qquad J = 1 + \alpha^2 > 0 \tag{6.30}$$

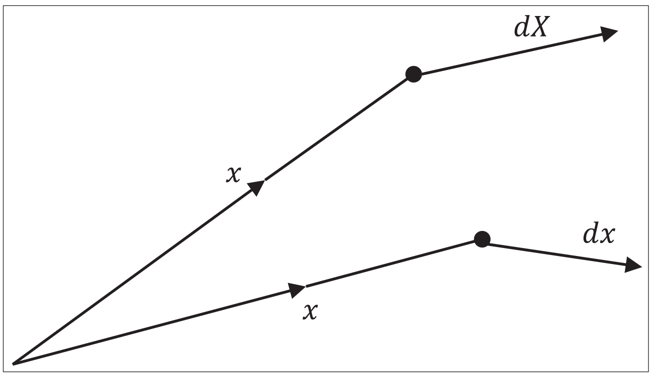

Figure 2: Deformation as a sequence of stretches and a rotation

Hence,

$$C = F^T F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & -\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & \alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\alpha^2 & 0 \\ 0 & 0 & 1+\alpha^2 \end{pmatrix} \tag{3.31}$$

and therefore

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(1+\alpha^2\right)^{\frac{1}{2}} & 0 \\ 0 & 0 & \left(1+\alpha^2\right)^{\frac{1}{2}} \end{pmatrix} \tag{3.32}$$

It can easily be shown likewise that $B = C$ and $V = U$. We may calculate R from the relation $R = FU^{-1}$. Thus,

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\alpha \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(1+\alpha^2\right)^{\frac{1}{2}} & 0 \\ 0 & 0 & \left(1+\alpha^2\right)^{\frac{1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(1+\alpha^2\right)^{\frac{1}{2}} & -\alpha\left(1+\alpha^2\right)^{-\frac{1}{2}} \\ 0 & \alpha\left(1+\alpha^2\right)^{-\frac{1}{2}} & \left(1+\alpha^2\right)^{-\frac{1}{2}} \end{pmatrix} \tag{3.33}$$

Now, let $\alpha = \tan\theta\left(0 < \theta < \frac{1}{2}\pi\right)$, then

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

which represents a rotation through an angle θ about the 1-axis, using the usual corkscrew convention for the sign of the angle. Thus, the deformation may be accomplished by first performing stretches of magnitudes $\left(1+\alpha^2\right)^{\frac{1}{2}}$ in the two and three-directions and then a rotation about the 1-axis. Since in this example $B = C$ and $V = U$, these operations may be reversed in order.

If a portion of the body moves in such a manner that the distances between every pair of particles remain constant that portion is said to move as a rigid body. For such a motion no stretching of line elements occurs and so at each particle of the given portion,

$$B = C = U = V = I, \quad F = R \tag{3.34}$$

In general, of course, the motion of the body does produce changes in the lengths of line elements and analysis of these length changes leads us to an alternative interpretation of C and B. Suppose that dL and dl denote the lengths of the vector line elements dX and dx, respectively. Then, using (3.18) and (3.26) and the Kronecker delta (defined by $\delta_{KL} = 0, K \neq L, K = L$),

$$\left(dl\right)^2 - \left(dL\right)^2 = dx_i dx_i - dX_K dX_K$$

$$= x_{i,K} x_{i,L} dX_K dX_L - dX_K dX_K$$

$$= \left(C_{KL} - \delta_{KL} \right) dX_{KL} dX_L \tag{3.35}$$

and so the tensor C enables us to calculate the difference between the squared elements of length in the reference and current configurations. Alternatively, if we define the inverse deformation gradients using (3.5) as

$$X_{K,i} = \frac{\partial}{\partial x_i} \chi_K^{-1}(x,t) \tag{3.38}$$

then, since $X_{K,i} x_{i,A} = \delta_{KA}$ by chain rule of partial differentiation, it follows from (6.18) that

$$dX_K = X_{K,i} dx_i \tag{3.39}$$

Hence, we may write

$$(dl)^2 - (dL)^2 = dx_i dx_i - X_{K,i} X_{K,j} dx_i dx_i$$

It can easily be verified, using (3.26) and the result $\left(F^T\right)^{-1} = \left(F^{-1}\right)^T$, that

$$\left(F^{-1}\right)^T F^{-1} = B^{-1}, \qquad X_{K,i} X_{K,j} = B_{ij}^{-1} \tag{3.40}$$

and therefore

$$(dl)^2 - (dL)^2 = \left(\delta_{ij} - B_{ij}^{-1} \right) dx_i dx_j \tag{3.41}$$

The tensor B also provides us with a means of calculating the same difference of squared elements of length. As we have already noted, B and C are second-order symmetric tensors. Their principal axes and principal values are real and may be found in the usual manner (Spencer [1980] Sections 2.3 and 9.3). The characteristic equation for C is

$$\det\left(C_{KL} - \lambda \delta_{KL} \right) = 0$$

that is,

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0 \tag{3.42}$$

where

$$I_1 = C_{KK} = tr\, C$$

$$I_2 = \frac{1}{2}\left(C_{KK} C_{LL} - C_{KL} C_{KL} \right) = \frac{1}{2}\left(tr\, C\right)^2 - \frac{1}{2} tr\, C^2 \tag{3.43}$$

$$I_3 = \det C$$

and tr denotes the trace. The quantities I_1, I_2, I_3 are known as the principal invariants of C.
We also note here a useful physical interpretation of I_3. In view of the definitions (3.3) and (6.26)

$$I_3 = \det C = \left(\det F\right)^2 = J^2 \tag{3.44}$$

and if a given set of particles occupies an element of volume dV_0 in ℓ and dV in ℓ_t then using (6.21),

$$J = dV / dV_0 \tag{3.45}$$

Thus, recalling (6.4)

$$dV / dV_0 = \sqrt{I_3} \tag{3.46}$$

If no volume change occurs during the deformation, the deformation is said to be isochronic and

$$J = 1, \quad I_3 = 1 \tag{3.47}$$

The strain invariants are also of fundamental importance in the constitutive theory of elasticity and we also find the following relation useful:

$$I_2 = I_3 tr\left(B^{-1}\right) \tag{3.48}$$

To prove this, we first note that from the Cayley–Hamilton theorem (Spencer [1980] Section 2.4), a matrix satisfies its own characteristic equation. Since the principal invariants of B are identical to those of C, B must satisfy the equation

$$B^3 - I_1 B^2 + I_2 B - I_3 I = 0 \tag{3.49}$$

Now, B is non-singular, so multiplying (6.49) by B^{-1}, we find that

$$B^2 = I_1 B - I_2 I + I_3 B^{-1}$$

Taking the trace of this equation, we have

$$tr\left(B^2\right) = I_1^2 - 3I_2 + I_3 tr\left(B^{-1}\right) \tag{3.50}$$

And using (3.43), (3.50) reduces to (3.48).

Homogeneous deformation

A deformation of the form

$$x_i = A_{iK} X_K + a_i \tag{3.51}$$

in which A and a are constants, is known as a homogeneous deformation. Clearly, $F = A$ and $J = \det A$. Particularly, simple examples of such deformations are given below

i. Dilatation

Consider the deformation

$$x_1 = \alpha X_1,\ x_2 = \alpha X_2,\ x_3 = \alpha X_3 \tag{3.52}$$

where α is a constant, then

$$F = \alpha I,\ B = C = \alpha^2 I,\ J = \alpha^3 \tag{3.53}$$

and so, to satisfy (3.4), we must have $\alpha > 0$. The strain invariants (3.41) are easily seen to be

$$I_1 = 3\alpha^2,\ I_2 = 3\alpha^4,\ I_3 = \alpha^6 \tag{3.54}$$

In view of (3.45), we see that if $\alpha > 1$ the deformation represents an expansion; if $\alpha < 1$ the deformation becomes a contraction.

ii. Simple extension with lateral extension or contraction
Suppose that

$$x_1 = \alpha X_1,\ x_2 = \beta X_2,\ x_3 = \beta X_3 \tag{3.55}$$

Then,

$$F = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix},\ J = \alpha\beta^2 \tag{3.56}$$

and so $\alpha > 0$. If $\alpha > 1$, the deformation is a uniform extension in the 1-direction. If $\beta > 0$ the diagonal terms of F are all positive so that $U = F$ and $R = 1$; β measures the lateral extension $(\beta > 1)$, or contraction $(\beta < 1)$, in the 2,3-plane.
If $\beta < 0$ then

$$U = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\beta & 0 \\ 0 & 0 & -\beta \end{pmatrix},\ R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In this case, $-\beta$ measures the associated lateral extension or contraction and a rotation through an angle π about the 1-axis is included in the deformation.
If the material is incompressible, only isochoric deformation is possible in which case

$$J = \alpha\beta^2 = 1 \tag{3.57}$$

This means that $|\beta|$ is less than, or greater than, unity according to as to whether α is greater than, or less than unity. In other words, an extension in the 1-direction produces a contraction in the lateral directions and vice versa. The strain tensors are found to be

$$B = C = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^2 \end{pmatrix} \tag{3.58}$$

and the invariants are

$$I_1 = \alpha^2 + 2\beta^2,\ I_2 = 2\alpha^2\beta^2 + \beta^4,\ I_3 = \alpha^2\beta^4 \tag{3.59}$$

Example 3.2:
The previous two deformations are special cases of

$$x_1 = \lambda_1 X_1,\ x_2 = \lambda_2 X_2,\ x_3 = \lambda_3 X_3 \tag{3.60}$$

Where $\lambda_i (i = 1,2,3)$ are constants. Show that, for the deformation (3.60) to satisfy $J > 0$, at least one of the λ_i has to be positive. Interpret the deformation geometrically in the case when all the λ_i are positive and show that in all cases the principal invariants are

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,\ I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2,\ I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \tag{3.61}$$

iii. Simple shear
Consider the deformation

$$x_1 = X_1 + \kappa X_2,\ x_2 = X_2,\ x_3 = X_3 \tag{3.62}$$

where κ is a constant. The particles move only in the 1-direction, and their displacement is proportional to their 2-coordinate. This deformation is known as a simple shear. Plane parallel to $X_1 = 0$ is rotated about an axis parallel to the three-axis through an angle $\theta = \tan^{-1}\kappa$, known as the angle of shear. The sense of the rotation is indicated in Figure 3. The planes $X_3 =$ constant are called shearing planes; and lines parallel to X_3-axis are known axes of shear. The deformation-gradient tensor is

$$F = \begin{pmatrix} 1 & \kappa & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.63}$$

and the Cauchy-Green strain tensors are

$$C = F^T F = \begin{pmatrix} 1 & \kappa & 0 \\ \kappa & 1+\kappa^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = FF^T = \begin{pmatrix} 1+\kappa^2 & \kappa & 0 \\ \kappa & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.64}$$

The tensor B^{-1} may be found using the formula

$$B^{-1} = \operatorname{adj} B / \det B \tag{3.65}$$

where $\operatorname{adj} B$ denotes the adjoint matrix of B. Thus

$$B^{-1} = \begin{pmatrix} 1 & -\kappa & 0 \\ -\kappa & 1+\kappa^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.66}$$

The strain invariants are

$$I_1 = 3+\kappa^2, \; I_2 = 3+\kappa^2, \; I_3 = 1 \tag{3.67}$$

Non-homogeneous deformations

Deformations which are not of the form (3.51) are referred to as non-homogenous deformations. We now discuss two such deformations which may be applied to either a solid or hollow circular cylinder. In each case, we take our coordinate system such that the X_3-axis coincides with the axis of the cylinder and the base lies in the plane $X_3 = 0$.

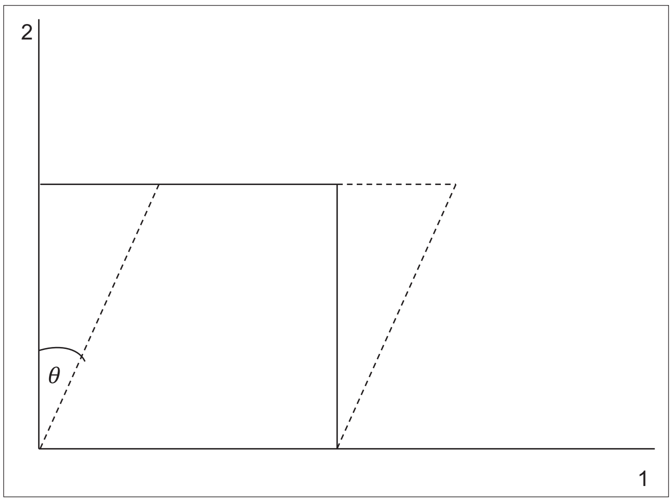

Figure 3: Non-homogenous deformations

i. Simple torsion

Consider the deformation in which each cross-section remains in its original plane but is rotated through an angle τX_3 about the three-axis, where τ is a constant called the twist per unit length (see 3.3). This deformation is referred to as simple torsion. Since each cross-section remains in its original plane,

$$x_3 = X_3 \tag{3.68}$$

To find the remaining equations which specify this deformation, we consider the typical cross-section, as shown in Figure 4, which is at a distance X_3 from the base of the cylinder. If P is the particle whose initial coordinates are $\left(X_1, X_2\right)$ then we may write

$$X_1 = R\cos\alpha,\ X_2 = R\sin\alpha \tag{3.69}$$

where $R = \left(X_1^2 + X_2^2\right)^{\frac{1}{2}}$. After the deformation this particle occupies the point p with coordinates $\left(x_1, x_2\right)$ where from the figure it follows that

$$x_1 = R\cos\left(\tau X_3 + \alpha\right),\ x_2 = R\sin\left(\tau X_3 + \alpha\right) \tag{3.70}$$

Expanding the sine and cosine functions and using (3.69) we obtain

$$x_1 = cX_1 - sX_2,\ x_2 = sX_1 + xX_2 \tag{3.71}$$

where $c = \cos\tau X_3,\ s = \sin\tau X_3$. For the deformation specified by (3.68) and (3.72), the deformation gradient is given by

$$F = \begin{pmatrix} c & -s & -\tau\left(sX_1 + cX_2\right) \\ s & c & \tau(cX_1 - sX_2) \\ 0 & 0 & 1 \end{pmatrix} \tag{3.72}$$

so that $J = 1$ and the deformation is isochoric. Further,

$$B = \begin{pmatrix} 1+\tau^2(sX_1 + cX_2) & -\tau^2(sX_1 + cX_2)(cX_1 - sX_2) & -\tau(sX_1 + cX_2) \\ -\tau^2(sX_1 + cX_2)(cX_1 - sX_2) & 1+\tau^2\left(cX_1 - sX_2\right)^2 & \tau(cX_1 - sX_2) \\ -\tau(sX_1 + cX_2) & \tau(cX_1 - sX_2) & 1 \end{pmatrix} \tag{3.73}$$

Since $J = 1, \det B = J^2 = 1$ and using (3.65) it follows that

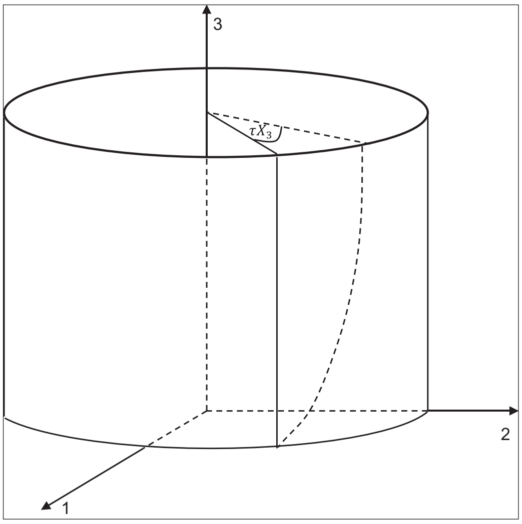

Figure 4: Simple torsion

$$B^{-1} = \begin{pmatrix} 1 & 0 & \tau(sX_1 + cX_2) \\ 0 & 1 & -\tau(cX_1 - sX_2) \\ \tau(sX_1 + cX_2) & -\tau(cX_1 - sX_2) & 1 + \tau^2(X_1^2 + X_2^2) \end{pmatrix} \tag{3.74}$$

Using (3.48), (3.72), and (3.73), we obtain

$$I_1 = I_2 = 3 + \tau^2 R^2 \tag{3.75}$$

ii. Torsion, extension, and inflation

Finally, we discuss the deformation which corresponds to simple extension along the axis of the cylinder, followed by simple torsion about its axis with twist τ per unit length. As a result of a uniform extension along the axis the particle X is displaced to X', where

$$X_1' = \beta X_1,\ X_2' = \beta X_2,\ X_3' = \alpha X_3 \tag{3.76}$$

and here we take $\alpha > 0, \beta > 0.$ If we now apply simple torsion to the extended cylinder, using (3.68) and (3.71), the final position x of the particle X is given by

$$x_1 = X_1' \cos\left(\tau X_3'\right) - X_2' \sin\left(\tau X_3'\right),$$

$$x_2 = X_1' \sin\left(\tau X_3'\right) + X_2' \cos\left(\tau X_3'\right) \tag{3.77}$$

$$x_3 = X_3'$$

Combining (3.76) and (3.77), we obtain the deformation

$$x_1 = \beta\left\{X_1 \cos\left(\alpha\tau X_3\right) - X_2 \sin\left(\alpha\tau X_3\right)\right\},$$

$$x_2 = \beta\left\{X_1 \sin\left(\alpha\tau X_3\right) + X_2 \cos\left(\alpha\tau X_3\right)\right\}, x_3 = \alpha X_3 \tag{3.78}$$

As a result of this deformation, the length of the cylinder increases or decreases depending on whether $\alpha > 1$ or $\alpha < 1$. Furthermore, since

$$x_1^2 + x_2^2 = \beta^2\left(X_1^2 + X_2^2\right) \tag{3.79}$$

The radius of the cylinder increases if $\beta > 1$ and decreases if $\beta < 1$. The deformation is usually referred to as torsion, extension, and inflation. The deformation gradient is given by

$$F = \begin{pmatrix} \beta c & -\beta s & -\alpha\tau\beta(sX_1 + cX_2) \\ \beta s & \beta c & \alpha\tau\beta\left(cX_1 - sX_2\right) \\ 0 & 0 & \alpha \end{pmatrix} \tag{3.80}$$

where $s = \sin\alpha\tau X_3$, $c = \cos\alpha\tau X_3$, so that $J = \alpha\beta^2$. If the material is incompressible, only isochoric deformations are possible, in which case

$$\beta = \alpha^{-\frac{1}{2}}$$

Then

$$B=\begin{pmatrix} \alpha^{-1}+\alpha\tau^{2}\left(sX_1+cX_2\right)^{2} & -\alpha\tau^{2}\left(sX_1+cX_2\right)\left(cX_1-sX_2\right) & -\alpha^{\frac{3}{2}}\tau\left(sX_1+cX_2\right) \\ -\alpha\tau^{2}\left(cX_1-sX_2\right)\left(sX_1+cX_2\right) & \alpha^{-1}+\alpha\tau^{2}\left(cX_1-sX_2\right)^{2} & \alpha^{\frac{3}{2}}\tau\left(cX_1-sX_2\right) \\ -\alpha^{\frac{3}{2}}\tau\left(sX_1+cX_2\right) & \alpha^{\frac{3}{2}}\tau\left(cX_1-sX_2\right) & \alpha^{2} \end{pmatrix} \tag{3.81}$$

Since for isochoric deformation $J=1, \det B=1$ and

$$B^{-1}=\begin{pmatrix} \alpha & 0 & \alpha^{\frac{1}{2}}\tau(sX_1+cX_2) \\ 0 & \alpha & -\alpha^{\frac{1}{2}}\tau\left(cX_1-sX_2\right) \\ \alpha^{\frac{1}{2}}\tau(sX_1+cX_2) & -\alpha^{\frac{1}{2}}\tau\left(cX_1-sX_2\right) & \alpha^{-2}\{1+\alpha^{2}\tau^{2}(X_1^{2}\left(X_1^{2}+X_2^{2}\right) \end{pmatrix} \tag{3.82}$$

Also $I_3=1$ and from (3.81) and (3.82) using (3.48) it follows that

$$I_1=\alpha^{2}+2\alpha^{-1}\alpha\tau^{2}R^{2},\quad I_2=2\alpha+\alpha^{-2}+\tau^{2}R^{2} \tag{3.83}$$

Exact solutions for problems with boundary conditions

In this section, we investigate the possibility of finding exact solutions of the equations without restriction on the form of the strain-energy function except for that imposed in some cases by the incompressibility condition.

Basic equations, boundary conditions

Restricting our attention to bodies maintained in equilibrium, the remaining equations which have to be satisfied for a compressible material are

$$\rho J=\rho_0 \tag{3.2.1}$$

and

$$T_{ij,j}+\rho b_i=0 \tag{3.2.2}$$

where

$$T_{ij}=\chi_0\delta_{ij}+\chi_1 B_{ij}+\chi_{-1}B_{ij}^{-1} \tag{3.2.3}$$

and χ_0,χ_1,χ_{-1} are functions of I_1, I_2 and I_3.

One way to maintain a body in equilibrium is to apply suitable surface tractions on its boundary. This type of problem gives rise to traction boundary conditions involving the stress vector. Let us consider a few typical examples. When a body of arbitrary shape is held in equilibrium under the action of a hydrostatic pressure $P(>0)$ per unit area of the surface of the deformed configuration, as shown in Figure 5,

$$t\left(n\right)=-Pn \tag{3.2.4}$$

As another example, consider a block extended in the 1-direction and maintained in equilibrium by applying a uniform tension T per unit area of the deformed configuration on its end faces, as shown in Figure 6. Suppose that after the deformation the block occupies the region

$$-a \le x_1 \le a, -b \le x_2 \le b, -c \le x_3 \le c$$

The end faces $x_1 = \pm a$ are both perpendicular to the 1-direction but they have different outward unit normal. On the face, $x_1 = -a$ the outward unit normal is $-e_1$ so that our boundary condition specifies $t(-e_1)$ and requires that

$$t(-e_1) = -Te_1 \tag{3.2.5}$$

But at any point on this face, $t(-e_1) = -t(e_1) = -(T_{11}, T_{12}, T_{13})$, so that an equivalent statement is

$$T_{11} = T,\ T_{12} = T_{13} = 0 \tag{3.2.6}$$

Likewise, on the face $x_1 = a$,

$$t(e_1) = Te_1 \tag{3.2.7}$$

which again rise to (3.2.6). Since we are not applying any forces to the remaining faces, the applied surface traction on these faces is zero. Such boundaries are said to be traction free. We use this term rather than "stress free" since, in many cases, the surface is not "stress free" even though it is free of applied traction. The vanishing of the applied traction implies

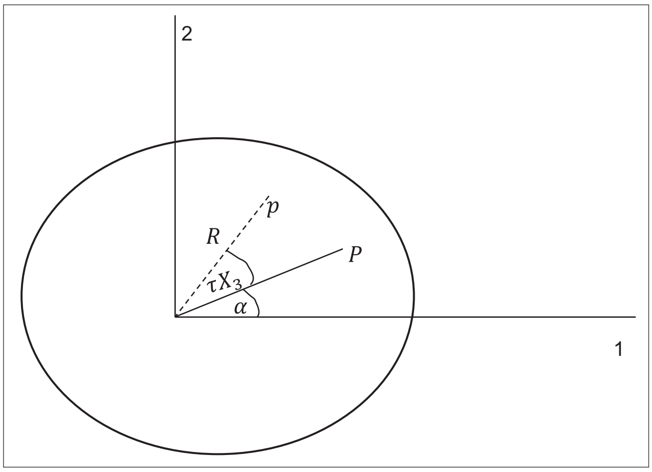

Figure 5: Equilibrium for a compressible material

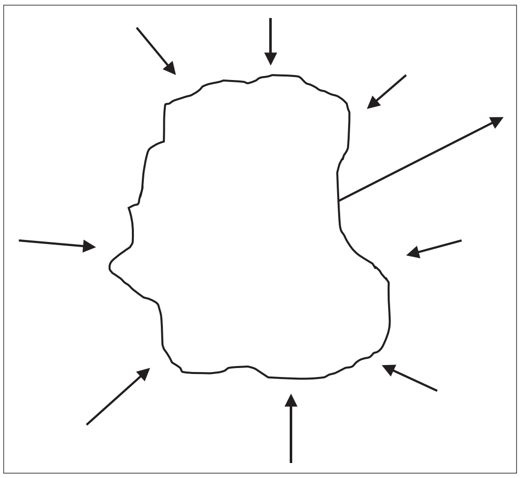

Figure 6: Applied traction

$$t\left(e_2\right)=0 \text{ on } x_2=b,\ t\left(-e_2\right)=0 \text{ on } x_2=-b \tag{3.2.8}$$

and so the stresses T_{21}, T_{22}, T_{23} are zero on $x_2=\pm b$, but in these faces $T_{11}\neq 0$. Similarly, the stresses T_{31}, T_{32}, T_{33} are zero on $x_3=\pm c$ but again $T_{11}\neq 0$.
Consider next a hollow circular cylinder, as shown in Figures 7 and 8, which is held in equilibrium under suitable surface tractions. In the deformed configuration, let the inner and outer radii be a_1 and a_2, respectively. The surface $r=a_1$ is subjected to a uniform pressure P and the outer surface $r=a_2$ is traction free. Traction is also applied to the end faces. On the surface, $r=a_1$ the outward unit normal is $-e_r$ so that our boundary condition on this surface specifies $t\left(-e_r\right)$ and requires that

$$t\left(-e_r\right)=Pe_r \tag{3.2.9}$$

on $r=a_1$. The outer surface is traction free provided

$$t\left(e_r\right)=0 \tag{3.2.10}$$

on $r=a_2$. Now $t_i=T_{ij}n_i$ and $e_r=r^{-1}\left(x_1,x_2,0\right)$, so that on $r=a_2$,

$$t_i\left(e_r\right)=T_{i\alpha}x_\alpha / a_2, \qquad \alpha=1,2 \tag{3.2.11}$$

Thus, a statement equivalent to (3.2.10) is

$$t_1=T_{11}\frac{x_1}{a_2}+T_{12}\frac{x_2}{a_2}=0, t_2=T_{21}\frac{x_1}{a_2}+T_{22}\frac{x_2}{a_2}=0, t_3=T_{31}\frac{x_1}{a_2}+T_{32}\frac{x_2}{a_2}=0 \tag{3.2.12}$$

The relation (3.2.9) may be expanded similarly.
We see from the above examples that in a particular problem, we are required to solve (3.2.1) and (3.2.2) subject to prescribed boundary conditions. In writing down these conditions, it is important to be able to identify the outward unit normal to the surface under consideration and also to realize which components

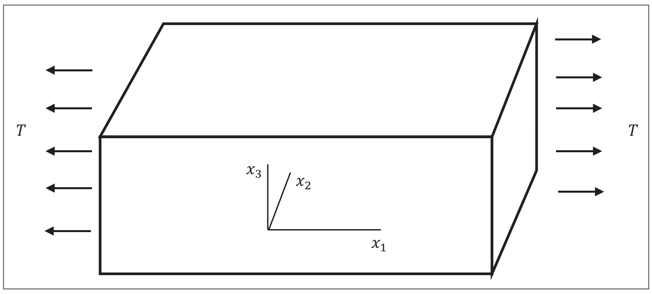

Figure 7: Traction free outer surface

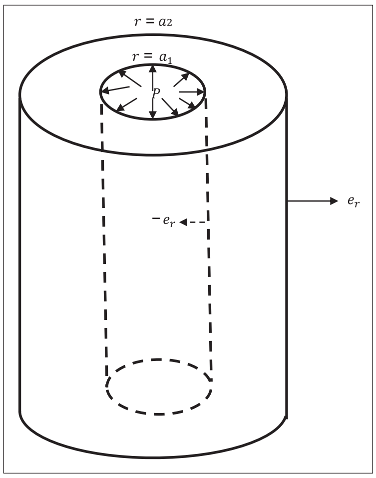

Figure 8: Outward unit normal to the surface

of the stress tensor are being specified by the applied surface tractions. In the above examples, the body is considered in the deformed configuration and T and P are measured per unit area in this state. It is, therefore, appropriate to use the Cauchy stress tensor T. In some situations, the applied forces may be measured per unit area of the reference configuration, in which case the Piola Kirchhoff stress tensors are more useful.
For an incompressible material, there is no volume change so that we have the condition

$$J = 1 \tag{3.2.13}$$

and the density has the value ρ_0 for all time. Equation (3.2.1) therefore reduces to an identity and the equations which have to be satisfied in this case are (3.2.13) and (3.2.2),

$$T_{ij} = -p\delta_{ij} + 2W_1 B_{ij} - 2W_2 B_{ij}^{-1} \tag{3.2.14}$$

p being an unknown scalar. As we shall see later, in a particular problem p is determined by equilibrium equations (3.2.2) and the specified boundary conditions.

REFERENCES

1. Siddiqi AH. Applied Functional Analysis. A Dekker Series of Monographs and Textbooks. New Brunswick, New Jersey: Anshan Publishers; 2008.
2. Eke A. Fundamental Analysis. Enugu, Nigeria: Acena Publishers; 1991.
3. Friedman A. Foundations of Modern Analysis. New York: Dover Publications Inc.; 1982.
4. Vatsa BS. Principles of Mathematical Analysis. New Delhi, India: CBS Publishers and Distributors; 2002.
5. Chidume CE. An iterative process for nonlinear lipschitzian strongly accretive mapping in spaces. J Math Anal Appl 1990;151:453-61.
6. Chidume CE. Approximation of fixed points of quasi-contractive mappings in spaces. Indian J Pure Appl Math 1991;22:273-81.
7. Chidume CE. Foundation of Mathematical Analysis. Trieste, Italy: The Abdusalam ICTP; 2006.
8. Chidume CE, Chidume CO. Foundations of Riemann Integration (Monograph). Trieste, Italy: The Abdusalam International Center for Theoretical Physics; 2003.
9. Chidume CE. Functional Analysis an Introduction to Metric Spaces. Nigeria: Longman; 1989.
10. Chidume CE, Lubuma MS. Solution of the stokes system by boundary integral equations and fixed point iterative schemes. J Niger Math Soc 1992;2:1-17.
11. Moore C. Lecture Notes on Advance Linear Functional Analysis. Awka, Nigeria: Nnamdi Azikiwe University; 2001.
12. Kreyzig E. Introductory Functional Analysis with Applications. New York: John Wiley and Sons; 1978.
13. Riez F, Nagy B. Functional Analysis. New York: Dover Publications, Inc.; 1990.
14. Royden HL. Real Analysis. New Delhi, India: Prentice Hall of India; 2008.
15. Argyros IK. Approximate Solution of Operator Equations with Applications. Singapore: World Scientific Publishing Co. PLC Ltd.; 2005.
16. Ukpong K. Undergraduate Real Analysis Lecture Notes. Yola, Nigeria: Federal University of Technology; 1987.
17. Altman M. Contractors and Contractor Directions Theory and Applications a New Approach to Solving Equations. New York and Basel: Marcel Dekker Inc.; 1977.
18. Atkin RJ, Fox N. An Introduction to the Theory of Elasticity. London and New York: Longman; 1980.
19. Bhat RB, Dukkipati RV. Advanced Dynamics. New Delhi: Narosa Publishing House; 2005.
20. Krantz SG. Real Analysis and Foundations. Boca Raton: Chapman and Hall CRC; 2010.
21. Ejike UB. A Postgraduate Lecture Note in Functional Analysis. Owerri, Nigeria: Federal University of Technology; 1991. p. 36.
22. Trench WF. Instructors Solution Manual Introduction to Real Analysis. Abuja, Nigeria: National Mathematical Center; 2010.
23. Trench WF. Introduction to Real Analysis. Abuja, Nigeria: National Mathematical Center; 2010.