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# Some Results on Common Fixed Point Theorems in Hilbert Space

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## ABSTRACT

This article provides the existence and uniqueness of a common fixed point for a pair of self-mappings, positive integers powers of a pair, and a sequence of self-mappings over a closed subset of a Hilbert space satisfying various contraction conditions involving rational expressions.

**Key words:** Hilbert space, closed subset, Cauchy sequence, completeness.
**Mathematics Subject Classification:** 40A05, 47H10, 54H25.

## INTRODUCTION

From the well-known Banach's contraction principle, every contraction mapping of a complete metric space into itself has a unique fixed point. This celebrated principle has played an important role in the development of different results related to fixed points and approximation theory. Later, it has been generalized by many authors[1-14] in metric space by involving either ordinary or rational terms in the inequalities presented there. Again, these results were extended by other or the same researchers[15-21] in different spaces. Thereafter, these results were again extended/developed for getting a unique fixed point to a pair of continuous self/different mappings[22-26] or more than two or a sequence of mappings, which may be continues or non-continues or imposing some weaker conditions on the mappings in the same or different spaces.[27-33]

As already stated, many authors have generalized Banach's contraction principle through rational expressions in different spaces such as metric, convex Banach's space, and normed spaces. But still here are some extensions/generalizations of this principle paying an attention in terms of taking rational expression in Hilbert spaces.

In this paper, we have considered a pair of self-mappings $T_1$ and $T_2$ of a closed subset $X$ of a Hilbert space, satisfying certain rational inequalities and obtained a unique fixed point for both $T_1$ and $T_2$. We have developed this result to the pair $T_1^p$ and $T_2^q$ where $p$ and $q$ are some positive integers and then also further extended the same results to a sequence of mappings. In each of the above three cases, we have obtained common fixed point in $X$. These results are generalizations of Koparde and Waghmode[29] in Hilbert space.

## MAIN RESULTS

**Theorem 1:** Let $T_1$ and $T_2$ be two self-mappings of a closed subset $X$ of a Hilbert space satisfying the inequality

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$$\|T_1x-T_2y\|^2 \le \alpha\frac{\|x-T_2y\|^2\left[1+\|y-T_1x\|^2\right]}{1+\|x-y\|^2}+\beta\frac{\|x-y\|^2\left[1+\|x-T_2y\|^2\right]}{1+\|x-y\|^2}$$
$$+\gamma\left[\|y-T_1x\|^2+\|x-T_2y\|^2\right]+\delta\|x-y\|^2$$

for all $x, y \in X$ and $x \neq y$, where $\alpha,\beta,\gamma,\delta$ are positive real's with $4\alpha+\beta+4\gamma+\delta<1$. Then, $T_1$ and $T_2$ have a unique common fixed point in $X$.

**Proof:** Let us start with an arbitrary point $x_0 \in X$, define a sequence $\{x_n\}$ as

$x_1 = T_1x_0$, $x_2 = T_2x_1$, $x_3 = T_1x_2$,.........

In general,

$x_{2n+1} = T_1x_{2n}$, $x_{2n+2} = T_2x_{2n+1}$, for $n = 0,1,2,3,$........

Next, we have to show that this sequence $\{x_n\}$ is a Cauchy sequence in $X$. For this consider

$$\|x_{2n+1}-x_{2n}\|^2 = \|T_1x_{2n}-T_2x_{2n-1}\|^2$$
$$\le \alpha\frac{\|x_{2n}-T_2x_{2n-1}\|^2\left[1+\|x_{2n-1}-T_1x_{2n}\|^2\right]}{1+\|x_{2n}-x_{2n-1}\|^2}+\beta\frac{\|x_{2n}-x_{2n-1}\|^2\left[1+\|x_{2n}-T_2x_{2n-1}\|^2\right]}{1+\|x_{2n}-x_{2n-1}\|^2}$$
$$+\gamma\left[\|x_{2n-1}-T_1x_{2n}\|^2+\|x_{2n}-T_2x_{2n-1}\|^2\right]+\delta\|x_{2n}-x_{2n-1}\|^2$$

$$\Rightarrow \|x_{2n+1}-x_{2n}\|^2 \le \frac{\beta+2\gamma+\delta}{1-2\gamma}\|x_{2n}-x_{2n-1}\|^2$$

A similar calculation indicates that

$$\|x_{2n+2}-x_{2n+1}\|^2 \le \frac{(2\alpha+\beta+\gamma)+(2\beta+\delta)\|x_{2n+1}-x_{2n}\|^2}{(1-2\alpha)+(1-2\beta)\|x_{2n+1}-x_{2n}\|^2}\|x_{2n+1}-x_{2n}\|^2$$

Let $k=\dfrac{\beta+2\gamma+\delta}{1-2\gamma}$ and $s(n)=\dfrac{(2\alpha+\beta+\gamma)+(2\beta+\delta)\|x_{2n+1}-x_{2n}\|^2}{(1-2\alpha)+(1-2\beta)\|x_{2n+1}-x_{2n}\|^2}$, where $s(n)$ depends $n$. Since $4\alpha+\beta+4\gamma+\delta<1$, we see that $k<1$ and $s(n)<1$, for all $n$.

Let $S=\sup\{s(n): n=0,1,2,......\}$ and let $\lambda^2=\max\{k,S\}$ so that $0<\lambda<1$, as a result of which we get

$$\|x_{n+1}-x_n\| \le \lambda\ \|x_n-x_{n-1}\|$$

Repeating the above process in a similar manner, we get

$$\|x_{n+1}-x_n\| \le \lambda^n\|x_1-x_0\|\ ,n\ge 1$$

On taking $n\to\infty$, we obtain $\|x_{n+1}-x_n\|\to 0$. Hence, it follows that the sequence $\{x_n\}$ is a Cauchy sequence. However, $X$ is a closed subset of Hilbert space and so by the completeness of $X$, there is a point $\mu \in X$ such that

$x_n \to \mu$ as $n\to\infty$.

Consequently, the sequences $\{x_{2n+1}\} = \{T_1 x_{2n}\}$ and $\{x_{2n+2}\} = \{T_2 x_{2n+1}\}$ converge to the same limit $\mu$. Now, we shall show that this $\mu$ is a common fixed point of both $T_1$ and $T_2$.

For this, in view of the hypothesis note that

$$\begin{aligned}
\|\mu - T_1\mu\|^2 &= \|(\mu - x_{2n+2}) + (x_{2n+2} - T_1\mu)\|^2 \\
&\le \|\mu - x_{2n+2}\|^2 + \alpha \frac{\|\mu - T_2 x_{2n+1}\|^2 \left[1 + \|x_{2n+1} - T_1\mu\|^2\right]}{1 + \|\mu - x_{2n+1}\|^2} \\
&\quad + \beta \frac{\|\mu - x_{2n+1}\|^2 \left[1 + \|\mu - T_2 x_{2n+1}\|^2\right]}{1 + \|\mu - x_{2n+1}\|^2} + \gamma \left[\|x_{2n+1} - T_1\mu\|^2 + \|\mu - T_2 x_{2n+1}\|^2\right] \\
&\quad + \delta \|\mu - x_{2n+1}\|^2 + 2\|\mu - x_{2n+2}\|\|x_{2n+2} - T_1\mu\|
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\|\mu - T_1\mu\|^2 \le (\gamma + \delta)\|\mu - T_1\mu\|^2,$$

$\Rightarrow T_1\mu = \mu$, since $\gamma + \delta < 1$,

Similarly, by considering

$$\|\mu - T_2\mu\|^2 = \|(\mu - x_{2n+1}) + (x_{2n+1} - T_2\mu)\|^2,$$

we get $T_2\mu = \mu$. Thus, $\mu$ is a common fixed point of $T_1$ and $T_2$.

Finally, to prove the uniqueness of a fixed point $\mu$, let $v (\mu \ne v)$ be another fixed point of $T_1$ and $T_2$. Then, from the inequality, we obtain

$$\begin{aligned}
\|\mu - v\|^2 = \|T_1\mu - T_2 v\|^2 &\le \alpha \frac{\|\mu - T_2 v\|^2 \left[1 + \|v - T_1\mu\|^2\right]}{1 + \|\mu - v\|^2} + \beta \frac{\|\mu - v\|^2 \left[1 + \|\mu - T_2 v\|^2\right]}{1 + \|\mu - v\|^2} \\
&\quad + \gamma \left[\|v - T_1\mu\|^2 + \|\mu - T_2 v\|^2\right] + \delta \|v - \mu\|^2
\end{aligned}$$

$\Rightarrow \|\mu - v\|^2 \le (\alpha + \beta + 2\gamma + \delta)\|\mu - v\|^2$

$\Rightarrow \mu = v$, since $\alpha + \beta + 2\gamma + \delta < 1$.

Hence, $\mu$ is a unique common fixed point of both $T_1$ and $T_2$ in $X$.

This completes the proof of the theorem.

**Theorem 2:** Let $X$ be a closed subset of a Hilbert space and $T_1$, $T_2$ be two mappings on $X$ itself satisfying

$$\begin{aligned}
\|T_1^p x - T_2^q y\|^2 &\le \alpha \frac{\|x - T_2^q y\|^2 \left[1 + \|y - T_1^p x\|^2\right]}{1 + \|x - y\|^2} + \beta \frac{\|x - y\|^2 \left[1 + \|x - T_2^q y\|^2\right]}{1 + \|x - y\|^2} \\
&\quad + \gamma \left[\|y - T_1^p x\|^2 + \|x - T_2^q y\|^2\right] + \delta \|x - y\|^2
\end{aligned}$$

for all $x, y \in X$ and $x \ne y$, where $\alpha, \beta, \gamma, \delta$ are positive real's with $4\alpha + \beta + 4\gamma + \delta < 1$ and $p, q$ are positive integers. Then, $T_1$ and $T_2$ have a unique common fixed point in $X$.

**Proof:** From above Theorem-1, $T_1^p$ and $T_2^q$ have a unique common fixed point $\mu \in X$ so that $T_1^p \mu = \mu$ and $T_2^q \mu = \mu$.

Now, $T_1^p (T_1 \mu) = T_1 (T_1^p \mu) = T_1 \mu$

$\Rightarrow T_1 \mu$ is a fixed point of $T_1^p$.

But $\mu$ is a unique fixed point of $T_1^p$.

$\therefore T_1 \mu = \mu$

Similarly, we get $T_2 \mu = \mu$

$\therefore$ $\mu$ is a common fixed point of $T_1$ and $T_2$.

For uniqueness, let $v$ be another fixed point of $T_1$ and $T_2$ so that $T_1 v = T_2 v = v$. Then,

$$\|\mu - v\|^2 = \|T_1^p \mu - T_2^q v\|^2 \le \alpha \frac{\|\mu - T_2^q v\|^2 \left[1 + \|v - T_1^p \mu\|^2\right]}{1 + \|\mu - v\|^2} + \beta \frac{\|\mu - v\|^2 \left[1 + \|\mu - T_2^q v\|^2\right]}{1 + \|\mu - v\|^2}$$
$$+ \gamma \left[\|v - T_1^p \mu\|^2 + \|\mu - T_2^q v\|^2\right] + \delta \|v - \mu\|^2$$

$$\Rightarrow \|\mu - v\|^2 \le (\alpha + \beta + 2\gamma + \delta) \|\mu - v\|^2$$

$\Rightarrow \mu = v$, since $\alpha + \beta + 2\gamma + \delta < 1$

Hence, $\mu$ is a unique common fixed point of $T_1$ and $T_2$ in $X$.
Complete the proof of the theorem.
In the following theorem, we have taken a sequence of mappings on a closed subset of a Hilbert space converges pointwise to a limit mapping and show that if this limit mapping has a fixed point; then, this fixed point is also the limit of fixed points of the mappings of the sequence.

**Theorem 3:** Let $X$ be a closed subset of a Hilbert space and let $\{T_i\}$ be a sequence of mappings on $X$ into itself converging pointwise to $T$ satisfying the following condition

$$\|T_i x - T_i y\|^2 \le \alpha \frac{\|x - T_i y\|^2 \left[1 + \|y - T_i x\|^2\right]}{1 + \|x - y\|^2} + \beta \frac{\|x - y\|^2 \left[1 + \|x - T_i y\|^2\right]}{1 + \|x - y\|^2}$$
$$+ \gamma \left[\|y - T_i x\|^2 + \|x - T_i y\|^2\right] + \delta \|x - y\|^2$$

for all $x, y \in X$ and $x \ne y$, where $\alpha, \beta, \gamma, \delta$ are positive real's with $4\alpha + \beta + 4\gamma + \delta < 1$. If each $T_i$ has a fixed point $\mu_i$ and $T$ has a fixed point $\mu$, then the sequence $\{\mu_n\}$ converges to $\mu$.

**Proof:** Since $\mu_i$ is a fixed point of $T_i$, then we have

$$\|\mu - \mu_n\|^2 = \|T\mu - T_n \mu_n\|^2 = \|(T\mu - T_n \mu) + (T_n \mu - T_n \mu_n)\|^2$$
$$\le \|T\mu - T_n \mu\|^2 + \|T_n \mu - T_n \mu_n)\|^2 + 2\|T\mu - T_n \mu\| \|T_n \mu - T_n \mu_n)\|$$
$$= \|T\mu - T_n \mu\|^2 + \alpha \frac{\|\mu - T_n \mu_n\|^2 \left[1 + \|\mu_n - T_n \mu\|^2\right]}{1 + \|\mu - \mu_n\|^2} + \beta \frac{\|\mu - \mu_n\|^2 \left[1 + \|\mu - T_n \mu_n\|^2\right]}{1 + \|\mu - \mu_n\|^2}$$
$$+ \gamma \left[\|\mu_n - T_n \mu\|^2 + \|\mu - T_n \mu_n\|^2\right] + \delta \|\mu - \mu_n\|^2 + 2\|T\mu - T_n \mu\| \|T_n \mu - T_n \mu_n\|$$

which implies that

$$\|\mu-\mu_n\|^2 = \|T\mu - T_n\mu\|^2 + \alpha\frac{\|\mu-T_n\mu_n\|^2\left[1+\|\mu_n-T_n\mu\|^2\right]}{1+\|\mu-\mu_n\|^2}$$

$$+\beta\frac{\|\mu-\mu_n\|^2\left[1+\|\mu-T_n\mu_n\|^2\right]}{1+\|\mu-\mu_n\|^2}+\gamma\left[\|\mu_n-T_n\mu\|^2+\|\mu-T_n\mu_n\|^2\right]$$

$$+\delta\|\mu-\mu_n\|^2+2\|T\mu-T_n\mu\|\|T_n\mu-T_n\mu_n\|$$

Letting $n\to\infty$ so that $T_n\mu\to T\mu$, $2\|T\mu-T_n\mu\|\|T_n\mu-T_n\mu_n\|\to 0$, we get

$$\lim_{n\to\infty}\|\mu-\mu_n\|^2 \le (\alpha+\beta+2\gamma+\delta)\lim_{n\to\infty}\|\mu-\mu_n\|^2$$

$$\Rightarrow \quad \lim_{n\to\infty}\|\mu-\mu_n\| = 0, \text{ since } \alpha+\beta+2\gamma+\delta<1$$

$$\Rightarrow \quad \mu_n\to\mu \text{ as } n\to\infty$$

This completes the proof.

The following theorem is the refinement of the above Theorem-1, by involving three rational terms in the inequality.

**Theorem 4:** Let $X$ be a closed subset of a Hilbert space and $T_1$, $T_2$ be two self-mappings on $X$ satisfying the following condition, then $T_1$ and $T_2$ have a unique common fixed point in $X$.

$$\|T_1x-T_2y\|^2 \le \alpha\frac{\|x-T_2y\|^2\left[1+\|y-T_1x\|^2\right]}{1+\|x-y\|^2}+\beta\frac{\|y-T_1x\|^2\left[1+\|x-T_2y\|^2\right]}{1+\|x-y\|^2}$$

$$+\gamma\frac{\|x-y\|^2\left[1+\|x-T_1x\|^2\right]}{1+\|y-T_2y\|^2}+\delta\|x-y\|^2$$

for all $x, y\in X$ and $x\neq y$, where $\alpha,\beta,\gamma,\delta$ are positive real's with $4(\alpha+\beta)+\gamma+\delta<1$.

**Proof:** Let us construct a sequence $\{x_n\}$ for an arbitrary point $x_0\in X$ as follows:

$x_{2n+1}=T_1x_{2n}$, $x_{2n+2}=T_2x_{2n+1}$ for $n=0,1,2,3,........$

Then

$$\|x_{2n+1}-x_{2n}\|^2 = \|T_1x_{2n}-T_2x_{2n-1}\|^2$$

$$\le \alpha\frac{\|x_{2n}-T_2x_{2n-1}\|^2\left[1+\|x_{2n-1}-T_1x_{2n}\|^2\right]}{1+\|x_{2n}-x_{2n-1}\|^2}+\beta\frac{\|x_{2n-1}-T_1x_{2n}\|^2\left[1+\|x_{2n}-T_2x_{2n-1}\|^2\right]}{1+\|x_{2n}-x_{2n-1}\|^2}$$

$$+\gamma\frac{\|x_{2n}-x_{2n-1}\|^2\left[1+\|x_{2n}-T_1x_{2n}\|^2\right]}{1+\|x_{2n-1}-T_2x_{2n-1}\|^2}+\delta\|x_{2n}-x_{2n-1}\|^2$$

which suggests that

$$\|x_{2n+1}-x_{2n}\|^2 \le s(n)\|x_{2n}-x_{2n-1}\|^2, \text{ where } s(n)=\frac{(2\beta+\gamma+\delta)+\delta\|x_{2n}-x_{2n-1}\|^2}{(1-2\beta)+(1-\gamma)\|x_{2n}-x_{2n-1}\|^2}$$

Similarly, we get

$$\|x_{2n+2}-x_{2n+1}\|^2 \le t(n)\|x_{2n+1}-x_{2n}\|^2, \text{ where } t(n)=\frac{(2\alpha+\gamma+\delta)+(\gamma+\delta)\|x_{2n+1}-x_{2n}\|^2}{(1-2\alpha)+\|x_{2n+1}-x_{2n}\|^2}$$

Since $4(\alpha+\beta)+\gamma+\delta<1$ then both $s(n)<1$ and $t(n)<1$, for all $n$.

Let $S=\sup\{s(n):n=1,2,......\}$ and $T=\sup\{t(n):n=1,2,......\}$.

Let $\lambda^2=\max\{S,T\}$ so that $0<\lambda<1$. Hence, from above, we get

$$\|x_{n+1}-x_n\|\le\lambda\ \|x_n-x_{n-1}\|$$

Continuing the above process, we obtained

$$\|x_{n+1}-x_n\|\le\lambda^n\|x_1-x_0\|\ \ ,n\ge1$$

Hence, the sequence $\{x_n\}$ is a Cauchy sequence in $X$ and so it converges to a limit in $X$. Since the sequences $\{x_{2n+1}\}=\{T_1x_{2n}\}$ and $\{x_{2n+2}\}=\{T_2x_{2n+1}\}$ are subsequences of $\{x_n\}$, then $\{T_1x_{2n}\}$ and $\{T_2x_{2n+1}\}$ converges to the same $\mu$. Now to see that this $\mu$ is a common fixed point of $T_1$ and $T_2$. For this using the inequality, we arrive at

$$\begin{aligned}\|\mu-T_1\mu\|^2&=\|(\mu-x_{2n+2})+(x_{2n+2}-T_1\mu)\|^2\\&\le\|\mu-x_{2n+2}\|^2+\alpha\frac{\|\mu-T_2x_{2n+1}\|^2\left[1+\|x_{2n+1}-T_1\mu\|^2\right]}{1+\|\mu-x_{2n+1}\|^2}\\&+\beta\frac{\|x_{2n+1}-T_1\mu\|^2\left[1+\|\mu-T_2x_{2n+1}\|^2\right]}{1+\|\mu-x_{2n+1}\|^2}+\gamma\frac{\|\mu-x_{2n+1}\|^2\left[1+\|\mu-T_1\mu\|^2\right]}{1+\|x_{2n+1}-T_2x_{2n+1}\|^2}\\&+\delta\|\mu-x_{2n+1}\|^2+2\|\mu-x_{2n+2}\|\|x_{2n+2}-T_1\mu\|\end{aligned}$$

On taking $n\to\infty$, we obtained $\|\mu-T_1\mu\|^2\le\beta\|\mu-T_1\mu\|^2$, because $0<\beta<1$, it follows immediately that $T_1\mu=\mu$.

Similarly, we get $T_2\mu=\mu$ by considering,

$$\|\mu-T_2\mu\|^2=\|(\mu-x_{2n+1})+(x_{2n+1}-T_2\mu)\|^2$$

Thus, $\mu$ is a common fixed point of $T_1$ and $T_2$.

Next, we want to show that $\mu$ is a unique fixed point of $T_1$ and $T_2$. Let us suppose that $v(\mu\ne v)$ is also a common fixed point of $T_1$ and $T_2$. Then, in view of hypothesis, we have

$$\begin{aligned}\|\mu-v\|^2&\le\alpha\frac{\|\mu-T_2v\|^2\left[1+\|v-T_1\mu\|^2\right]}{1+\|\mu-v\|^2}+\beta\frac{\|v-T_1\mu\|^2\left[1+\|\mu-T_2v\|^2\right]}{1+\|\mu-v\|^2}\\&+\gamma\frac{\|\mu-v\|^2\left[1+\|\mu-T_1\mu\|^2\right]}{1+\|v-T_2v\|^2}+\delta\|v-\mu\|^2\end{aligned}$$

$$\Rightarrow\|\mu-v\|^2\le(\alpha+\beta+\gamma+\delta)\|v-\mu\|^2$$

$$\Rightarrow\mu=v,\text{ since }\alpha+\beta+\gamma+\delta<1$$

It follows that $\mu=v$ and so the common fixed point is unique.

This completes the proof of the theorem.

**Theorem 5:** Let $T_1$ and $T_2$ be the self-mappings on a closed subset $X$ of a Hilbert space satisfying

$$\left\|T_1^p x - T_2^q y\right\|^2 \le \alpha \frac{\left\|x - T_2^q y\right\|^2\left[1 + \left\|y - T_1^p x\right\|^2\right]}{1 + \|x - y\|^2} + \beta \frac{\left\|y - T_1^p x\right\|^2\left[1 + \left\|x - T_2^q y\right\|^2\right]}{1 + \|x - y\|^2}$$
$$+ \gamma \frac{\|x - y\|^2\left[1 + \left\|x - T_1^p x\right\|^2\right]}{1 + \left\|y - T_2^q y\right\|^2} + \delta\|x - y\|^2$$

for all $x, y \in X$ and $x \neq y$, where $\alpha, \beta, \gamma, \delta$ are positive real's with $4(\alpha + \beta) + \gamma + \delta < 1$ and $p, q$ are positive integers. Then, $T_1$ and $T_2$ have a unique common fixed point in $X$.

**Proof:** From above Theorem-3, $T_1^p$ and $T_2^q$ have a unique common fixed point $\mu \in X$ so that $T_1^p \mu = \mu$ and $T_2^q \mu = \mu$.

Now, $T_1^p\left(T_1\mu\right) = T_1\left(T_1^p\mu\right) = T_1\mu$

$\Rightarrow$ $T_1\mu$ is a fixed point of $T_1^p$.

But $\mu$ is a unique fixed point of $T_1^p$.

$\therefore$ $T_1\mu = \mu$

Similarly, we get $T_2\mu = \mu$

$\therefore$ $\mu$ is a common fixed point of $T_1$ and $T_2$.

For uniqueness, let $v$ be another fixed point of $T_1$ and $T_2$, so we have $T_1 v = T_2 v = v$. Then,

$$\|\mu - v\|^2 = \left\|T_1^p \mu - T_2^q v\right\|^2 \le \alpha \frac{\left\|\mu - T_2^q v\right\|^2\left[1 + \left\|v - T_1^p \mu\right\|^2\right]}{1 + \|\mu - v\|^2} + \beta \frac{\left\|v - T_1^p \mu\right\|^2\left[1 + \left\|\mu - T_2^q v\right\|^2\right]}{1 + \|\mu - v\|^2}$$
$$+ \gamma \frac{\|\mu - v\|^2\left[1 + \left\|\mu - T_1^p \mu\right\|^2\right]}{1 + \left\|v - T_2^q v\right\|^2} + \delta\|v - \mu\|^2$$

$\Rightarrow \|\mu - v\|^2 \le (\alpha + \beta + \gamma + \delta)\|v - \mu\|^2$

$\Rightarrow \mu = v$, since $\alpha + \beta + \gamma + \delta < 1$

This completes the proof of the theorem.

In the following last theorem, we consider a sequence of mappings, which converges pointwise to a limit mapping and shows that if this limit mapping has a fixed point, then this fixed point is also the limit of fixed points of the mappings of the sequence.

**Theorem 6:** Let $\{T_i\}$ be a sequence of mappings of $X$ into itself converging pointwise to $T$ and let

$$\left\|T_i x - T_i y\right\|^2 \le \alpha \frac{\left\|x - T_i y\right\|^2\left[1 + \left\|y - T_i x\right\|^2\right]}{1 + \|x - y\|^2} + \beta \frac{\left\|y - T_i x\right\|^2\left[1 + \left\|x - T_i y\right\|^2\right]}{1 + \|x - y\|^2}$$
$$+ \gamma \frac{\|x - y\|^2\left[1 + \left\|x - T_i x\right\|^2\right]}{1 + \left\|y - T_i y\right\|^2} + \delta\|x - y\|^2$$

for all $x, y \in X$ and $x \neq y$, where $\alpha, \beta, \gamma, \delta$ are positive real's with $4(\alpha+\beta)+\gamma+\delta<1$. If each $T_i$ has a fixed point $\mu_i$ and $T$ has a fixed point $\mu$, then the sequence $\{\mu_n\}$ converges to $\mu$ in $X$.

**Proof:** We know that $\mu_i$ is a fixed point of $T_i$ then

$$\begin{aligned}
\|\mu-\mu_n\|^2 &= \|T\mu - T_n\mu_n\|^2 \\
&= \|(T\mu - T_n\mu) + (T_n\mu - T_n\mu_n)\|^2 \\
&\leq \|T\mu - T_n\mu\|^2 + \|T_n\mu - T_n\mu_n)\|^2 + 2\|T\mu - T_n\mu\|\|T_n\mu - T_n\mu_n)\| \\
&= \|T\mu - T_n\mu\|^2 + \alpha\frac{\|\mu - T_n\mu_n\|^2\left[1+\|\mu_n - T_n\mu\|^2\right]}{1+\|\mu-\mu_n\|^2} \\
&\quad + \beta\frac{\|\mu_n - T_n\mu\|^2\left[1+\|\mu - T_n\mu_n\|^2\right]}{1+\|\mu-\mu_n\|^2} + \gamma\frac{\|\mu - \mu_n\|^2\left[1+\|\mu - T_n\mu\|^2\right]}{1+\|\mu_n - T_n\mu_n\|^2} \\
&\quad + \delta\|\mu-\mu_n\|^2 + 2\|T\mu - T_n\mu\|\,\|T_n\mu - T_n\mu_n\|
\end{aligned}$$

Letting $n \to \infty$ so that $T_n\mu \to T\mu$, $2\|T\mu - T_n\mu\|\|T_n\mu - T_n\mu_n\| \to 0$, we get

$$\lim_{n\to\infty}\|\mu-\mu_n\|^2 \leq (\alpha+\beta+\gamma+\delta)\lim_{n\to\infty}\|\mu-\mu_n\|^2$$

$$\Rightarrow \lim_{n\to\infty}\|\mu-\mu_n\| = 0, \text{ since } \alpha+\beta+\gamma+\delta<1 \Rightarrow \mu_n \to \mu \text{ as } n\to\infty$$

This completes the proof.

## CONCLUSIONS

In this paper, Banach's contraction principle has been generalized to obtain a common fixed point for two, positive powers of two, and a sequence of self-mappings defined over a closed subset of a Hilbert space. The existence and uniqueness of a common fixed point for three different types of self-mappings are carried out in all six theorems with various inequalities involving square rational terms.

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