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On Some Geometrical Properties of Proximal Sets and Existence of Best Proximity Points

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ABSTRACT

The notion of proximal intersection property and diagonal property is introduced and used to establish some existence of the best proximity point for mappings satisfying contractive conditions.

Key words: Best proximity point, proximal sets, UC property, proximal intersection property, diagonal property
Mathematics Subject Classifications: MSC 2010, 47H09

INTRODUCTION

Let X be a non-empty set and f be a self-map of X. An element $x \in X$ is called a fixed point of f if $f(x) = x$. Fixed point theorems deal with sufficient conditions on X and f which ensure the existence of fixed points. Suppose the fixed point equation $f(x) = x$ does not possess a solution, then the natural interest is to find an element $x \in X$, such that x is in proximity to $f(x)$ in some sense. In other words, we would like to get a desirable estimate for the quantity $d(x, f(x))$.

It is natural that some mapping, especially non-self mappings defined on a metric space (X,d), do not necessarily possess a fixed point that is $d(x, f(x)) > 0$ for all $x \in X$. In such situations, it is reasonable to search for existence and uniqueness of the point $x \in X$ such that $d(x, f(x)) = 0$. In other words, one speculates to determine an approximate solution x that is optimal in the sense that the distance between x and $f(x)$ is minimum. Here, the point x is the best proximity point. That is $d(x, f(x)) = d(A,B)$ Where $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$.

Best proximity results is also interesting for the geometrical properties of the underlying space. In Suzuki *et al.*,[1] UC property was introduced to prove some existence results on best proximity point. In Raj and Eldred,[2] the author introduced p-property and proved strict convexity is equivalent to p-property.

We introduce proximal intersection property and diagonal property for a pair (A,B) where A and B are nonempty closed subsets of metric space. We show that every pair (A,B) of a real Hilbert space satisfies diagonal property. Then, these properties are used to establish the existence of best proximity point for mapping satisfying some contractive conditions introduced by Wong.[3]

PRELIMINARIES

In this section, we give some basic definitions and concepts that are related to the context of our main results.

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Definition 2.1[4]
Let A and B be nonempty subsets of a metric space (X,d). Then, (A,B) is said to satisfy property UC if the following holds: If x_n and x'_n are sequences in A and y_n is a sequence in B such that $\lim_n d(x_n,y_n)=d(A,B)$ and $\lim_n d(x'_n,y_n)=d(A,B)$, then $\lim_n d(x_n,x'_n)=0$ holds.

Definition 2.2
Let A and B be nonempty subsets of a metric space (X,d). Then, (A,B) is said to satisfy proximal intersection property if whenever $A_n \subset A$ and $B_n \subset B$ are a decreasing sequence of closed subsets such that $\delta(A_n,B_n)\to d(A,B)$. Then $\bigcap A_n=\{x\},\bigcap B_n=\{y\}$ with $d(x,y)=d(A,B)$.

Remark 2.1
$d(A,B)=d(\bar{A},\bar{B})$ and $\delta(A,B)=\delta(\bar{A},\bar{B})$ where $\delta(A,B)=\sup\{\| x-y\| . x\in A, y\in B\}$.

Definition 2.3[2]
Let X be a metric space and let $f:X\to X$. Then, d_f is the function on $X\times X$ defined by

$$d_f(x,y)=\inf\left\{d\left(f^n(x),f^n(y)\right):n\ge 1\right\},x,y\in X \tag{1}$$

Definition 2.4[3]
Let A and B be nonempty subsets of a metric space X. We shall use X_d to denote the set

$$\left\{r' : \textit{for any } s>r', d(x,y)-d(A,B)\in[r',s]\ \textit{for some } x\in A, y\in B\right\} \tag{2}$$

Remark 2.2
If $r'\in X_d$, then, there exists $x_n\in A, y_n\in B$ such that $d(x_n,y_n)-d(A,B)\to r'$. Also if $x\in A, y\in B$, then $d(x_n,y_n)-d(A,B)\in X_d$ and if $x_n\in A, y_n\in B$ such that $d(x_n,y_n)-d(A,B)\to r'$ then $r'\in X_d$.

Definition 2.5
Let (A,B) be proximal pair of a metric space X. Then, (A,B) is said to satisfy diagonal property if whenever $s_n,t_n\in A$ and $s'_n,t'_n\in B$ are bounded sequences such that $d(s_n,s'_n)\to d(A,B)$ and $d(t_n,t'_n)\to d(A,B)$ then $d(s_n,t'_n)-d(s'_n,t_n)\to 0$.

Lemma 2.1[1]
Let A and B be nonempty subsets of a metric space (X,d). Then, (A,B) has the property UC. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B, respectively, such that either of the following holds:

$$\lim_{m\to\infty} sup_{n\ge m} d(x_m,y_n)=d(A,B)\, or$$

$$\lim_{n\to\infty} sup_{m\ge n} d(x_n,y_n)=d(A,B)$$

Then $\{x_n\}$ is Cauchy.

MAIN RESULTS

Theorem 3.1
Let A and B be nonempty closed subsets of a complete metric space X satisfying UC property. Let A_n,B_n be decreasing sequence of nonempty closed subsets of X such that $\delta(A_n,B_n)\to d(A,B)$ as $n\to\infty$. Then, $\bigcap A_n=\{x\},\bigcap B_n=\{y\}$ with $d(x,y)=d(A,B)$ that is (A,B) satisfies proximal intersection property.

Proof
Construct a sequence x_n,y_n in X by selecting $x_n\in A_n, y_n\in B_n$ for each $n\in N$.
Since $A_{n+1}\subseteq A_n, B_{n+1}\subseteq B_n$ for all n, we have $x_n\in A_n\subseteq A_m, y_n\in B_n\subseteq B_m$ for all $n>m$.
We claim that x_n is a Cauchy sequence.
Let $\in>0$ be given.
Since $\delta(A_n,B_n)\to d(A,B)$, there exists a positive integer N such that $\delta(A_n,B_n)<d(A,B)+\in$, for all

$n \geq N$.
Since A_n, B_n are decreasing sequences, we have $A_n, A_m \subseteq A_N$ and $B_n, B_m \subseteq B_N$ for all $m,n \geq N$.
Therefore, $x_n, x_m \in A_N$ and $y_n, y_m \in B_N$ for all $m,n \geq N$, and thus we have

$$d\left(x_n, x_m\right) \leq \delta\left(A_n, B_n\right) < d\left(A, B\right) + \in \text{ for all } m,n \geq N \tag{3}$$

Since A and B satisfy UC property from lemma 2.1 x_n is a Cauchy sequence. There exists $x \in A$ such that $x_n \to x$. Similarly, there exists $y \in B$ such that $y_n \to y$.
We claim that $x \in \bigcap A_n, y \in \bigcap B_n$.
Since A_n and B_n are closed for each n, $x \in A_n, y \in B_n$ for all $n \in N$.
Since $d\left(x_n, y_n\right) \to d\left(A, B\right)$ we have $d\left(x, y\right) = d(A, B)$.
Finally to establish that x is the only point in $\bigcap A_n$,
If $x_1 \neq x_2 \in \bigcap A_n$, then $d\left(x_i, y\right) = d\left(A, B\right)$
UC property forces $x_1 = x_2$. Similarly $\bigcap B_n = \{y\}$.

Lemma 3.1
Let A and B be nonempty closed convex subsets of a real Hilbert space. For every bounded sequence $u_n, v_n \in A$ and $u_n', v_n' \in B$, we have if $\left\|u_n - v_n\right\|$ and $\left\|u_n' - v_n'\right\| \to d(A,B)$ then
1) $\left(u_n - v_n\right) - (u_n' - v_n') \to 0$.
2) $\lim_{n\to\infty}\left(\left\|u_n - v_n'\right\| - \left\|v_n - u_n'\right\|\right) = 0.$

Proof
Let u_n, v_n be sequences in A and u_n', v_n' be sequences in B such that

$$\left\|u_n - u_n'\right\| \to d\left(A,B\right) \text{ and } \left\|v_n - v_n'\right\| \to d\left(A,B\right) \tag{4}$$

$$\text{Let } \varepsilon_n = \left\langle v_n' - u_n', u_n' - u_n \right\rangle \tag{5}$$

Since B is convex, $\lambda v_n' + (1-\lambda)u_n' \in B$ for all $0 \leq \lambda \leq 1$

$$\left\|u_n - v_n'\right\|^2 = \left\|u_n - u_n'\right\|^2 + \left\|u_n' - v_n'\right\|^2 + 2\left\langle u_n - u_n', u_n' - v_n' \right\rangle \tag{6}$$

$$\left\|v_n - u_n'\right\|^2 = \left\|v_n - v_n'\right\|^2 + \left\|v_n' - u_n'\right\|^2 + 2\left\langle v_n - v_n', v_n' - u_n' \right\rangle \tag{7}$$

$$\text{Using the identity } \frac{1}{4}\left(\left\|x+y\right\|^2 - \left\|x+y\right\|^2\right) = \left\langle x, y \right\rangle \tag{8}$$

Since $\left\|u_n - (\lambda v_n' + \left(1-\lambda\right)u_n')\right\| \geq d\left(A,B\right)$ for all n, $\limsup\left(\lambda\left\|u_n' - v_n'\right\|^2 + 2\varepsilon_n\right) \geq 0.$
Letting $\to 0$ $\limsup \varepsilon_n \geq 0$. Similarly, $\liminf \varepsilon_n \geq 0$.
Therefore,

$$\limsup\left\langle u_n - u_n', u_n' - v_n' \right\rangle \geq 0 \tag{9}$$

$$\liminf\left\langle u_n - u_n', u_n' - v_n' \right\rangle \geq 0 \tag{10}$$

Let $s_n = u_n - u_n'$ and $s_n' = v_n - v_n'$.
Suppose $s_n - s_n' \to 0$ there exists a subsequence n_k such that $\left\|s_{n_k} - s_{n_k}'\right\| \geq \in_0$ for some $\in_0$.
For this, $\in > 0$ there exists N such that for all $n_k \geq N$,

$$\left\|u_{n_k} - u_{n_k}'\right\| \leq d\left(A,B\right) + \in \tag{11}$$

$$\left\|v_{n_k} - v_{n_k}'\right\| \leq d\left(A,B\right) + \in \tag{12}$$

From the parallelogram law, for all $n_k \geq N$,

$$\left\|\frac{\left(u_{n_k} - u_{n_k}'\right) + \left(v_{n_k} - v_{n_k}'\right)}{2}\right\|^2 \leq \left\|\left(\frac{d\left(A,B\right) + \in}{2}\right)^2\right\| + \left\|\left(\frac{d\left(A,B\right) + \in}{2}\right)^2\right\| - \left(\frac{\in_0}{2}\right)^2 \tag{13}$$

As there exists $\in > 0$ such that the R.H.S. is strictly less than $\left(d\left(A,B\right)\right)^2$
a contradiction.
Therefore $\Rightarrow s_n - s_n' \to 0$
Let lim sup$\langle u_n - u_n', u_n' - v_n' \rangle$ = lim sup$\left(\langle s_n - s_n', u_n' - v_n' \rangle + \langle v_n - v_n', u_n' - v_n' \rangle\right) \geq 0$
As u_n and v_n are bounded sequences lim sup$\langle v_n - v_n', u_n' - v_n' \rangle \geq 0$ (14)
But lim sup$\langle v_n - v_n', v_n' - u_n' \rangle \geq 0$ analogous to (9) (15)

lim sup$- \langle v_n - v_n', v_n' - u_n' \rangle \geq 0$ from (14) (16)

$\Rightarrow -$lim $inf \langle v_n - v_n', v_n' - u_n' \rangle \geq 0$ (17)

Also lim $inf \langle v_n - v_n', v_n' - u_n' \rangle \geq 0$ is also true being analogous to (10)

$\Rightarrow$ lim $inf \langle v_n - v_n', v_n' - u_n' \rangle = 0$ (18)

Replacing lim inf and lim sup in the above arguments we have

lim sup$\langle v_n - v_n', v_n' - u_n' \rangle = 0$ (19)

Similarly

lim inf $\langle u_n - u_n', u_n' - v_n' \rangle = 0$ (20)

lim sup $\langle u_n - u_n', u_n' - v_n' \rangle = 0$ (21)

From 18,19 and 20,21 and from 6,7
We get the desired result, $\lim_{n\to\infty}\left(\|u_n - v_n'\| - \|v_n - u_n'\|\right) = 0$.

Lemma 3.2

Let A and B be non empty closed subsets of a complete metric space X such that (*A,B*) satisfying UC property. Let $f: A \cup B \to A \cup B$ be continuous. Suppose that $f(A) \subset B, f(B) \subset A$ be a be a continuous function such that

(a) $\inf\left\{d\left(x, f\left(x\right)\right): x \in A\right\} = d\left(A,B\right) = \inf\left\{d\left(x, f\left(x\right)\right): x \in A\right\} = d\left(A,B\right)$

(b) There exists $\delta_n > 0$ such that $d\left(f\left(x\right), f\left(y\right)\right) - d\left(A,B\right) < \frac{1}{n}$ whenever $max\left\{d\left(x, f\left(x\right)\right) - d\left(A,B\right), d\left(y, f\left(y\right)\right) - d(A,B)\right\} < \delta_n$ and $x \in A', y \in B'$ where A' and B' are any closed bounded sets of A and B, respectively.

Then, there exists a best proximity point $x \in A$ such that $d\left(x, f\left(x\right)\right) = d(A,B)$. Further, if $d\left(f(x), f\left(y\right)\right) = d(x,y)$ for all $x \in A, y \in B$ then the best proximity point is unique.

Proof

Let $A_n = \left\{x \in A : d\left(x, f\left(x\right)\right) - d\left(A,B\right) \leq \frac{1}{n}\right\}$

$B_n = \left\{y \in B : d\left(y, f\left(y\right)\right) - d\left(A,B\right) \leq \frac{1}{n}\right\}$

Since f is continuous, A_n and B_n are closed.
From (a) A_n and B_n are nonempty, there exits N for all $n \in N$.
Let $x \in A_n, y \in B_n$ then $d\left(x, f\left(x\right)\right) - d\left(A,B\right) < \delta_n$ and $d\left(y, f\left(y\right)\right) - d\left(A,B\right) < \delta_n$.
From (b) $d\left(f(x), f\left(y\right)\right) - d\left(A,B\right) \leq \frac{1}{n}$ where $\delta_n \to 0$.
For any $x \in A_n, y \in B_n$, $d\left(f(x), f\left(y\right)\right) - d\left(A,B\right) \leq \frac{1}{n}$
Which implies $\delta\left(f\left(A_n\right), f\left(B_n\right)\right) \to d(A,B)$

and hence $\delta\left(\overline{f(A_n)}, \overline{f(B_n)}\right) \to d(A,B)$.
By proximal intersection criterion for completeness
We have $\bigcap_{n\geq 1} \overline{f(A_n)} = y$, and $\bigcap_{n\geq 1} \overline{f(B_n)} = x$ and $d(x,y) = d(A,B)$.
Thus for each $n \geq 1$, there exists $x_n \in A_n$ such that $d(y, f(x_n)) < \frac{1}{n}$
Since $d(x_n, f(x_n)) \to d(A,B)$, and $d(y_n, f(y_n)) \to d(A,B)$.
By UC property $x_n \to x$.
Since A_n is closed, $x \in A_n$ for each n. This implies $d(x, f(x)) \to d(A,B)$.
Similarly, $y_n \to y$ such that $d(y, f(y)) \to d(A,B)$.
To prove uniqueness,

$d(x, f(x)) = d(A,B)$ and $d(x', f(x')) = d(A,B)$

Since f is non-expansive $d(f^2(x'), f(x')) = d(A,B)$
which implies $f^2(x') = x'$
As $d(x, f(x)) = d(f(x'), f^2(x')) = d(A,B$.
From (b) $d(f(x), x') = d(f(x), f^2(x')) = d(A,B$ which implies $x = x'$.

Theorem 3.2
Let A and B be nonempty closed subsets of a metric space X and let $f: A\bigcup B \to A\bigcup B$ be continuous such that $f(A) \subset B, f(B) \subset A$. Suppose that there exists $\phi: X_d \to [0,\infty]$ such that $d(x,y) - d(A,B) \leq \phi((x,y) - d(A,B))$ for all $x \in A, y \in B$ and $sup_{s>r} inf_{t\in[r,s]} (t - \phi(t)) > 0$ for $r \in X_d$-$\{0\}$. Then, $d_f(x,y) = d(A,B)$ for all $x \in A, y \in B$. Hence, $\inf\{d(x, f(x)) : x \in A\} = d(A,B)$.

Proof
Suppose to the contrary that there exists $x \in A, y \in B$ such that

$$\inf\left\{d(f^n(x), f^n(y)) : n \geq 1\right\} > d(A,B) \tag{22}$$

By hypothesis, there exists $s \in (r', \infty)$ such that

$u = inf_{t\in[r',s]} (t - \phi(t)) > 0$ where $r' = r - d(A,B)$.

Since there exists a sequence $d(f^n(x), f^n(y)) - d(A,B) \to r'$, where $r' \in X_d - \{0\}$.
Let $t \in (0, s-r')$, i.e. $t < s - r' \Rightarrow r' + t < s$.
Then, from (5), we have
$d(f^n(x), f^n(y)) - d(A,B) \to r' + t < s$ for some $n \geq 1$.

Since $d(f^n(x), f^n(y)) - d(A,B) \in [r', s]$

$$u \leq d(f^n(x), f^n(y)) - d(A,B) - \phi\left(d(f^n(x), f^n(y)) - d(A,B)\right)$$

$$\phi\left(d(f^n(x), f^n(y)) - d(A,B)\right) \leq d(f^n(x), f^n(y)) - d(A,B) - u \tag{23}$$

If $f^n(x) \in A, f^n(y) \in B$ and vice versa.
It follows that

$$d_f(x,y) - d(A,B) \leq d_f(f^n(x), f^n(y)) - d(A,B) \tag{24}$$

$$\leq d(f^n(x), f^n(y)) - d(A,B) \tag{25}$$

$$\leq \phi\left(d(f^n(x), f^n(y))\right) - d(A,B) \tag{26}$$

$$\leq d(f^n(x), f^n(y)) - d(A,B) \text{ from (23)} \tag{27}$$

$$< r' + t - u \tag{28}$$

Letting $t \to 0$, we have

$$d_f(x,y) - d(A,B) \le r' - u \tag{30}$$

$$d_f(x,y) - d(A,B) \le r - d(A,B) - u \tag{31}$$

$$d_f(x,y) \le r - u \tag{32}$$

a contradiction.

Theorem 3.3

Let A and B be nonempty closed subsets of a metric space X. Suppose (A,B) satisfies UC property and diagonal property. Let f be as in theorem 3.2, then f satisfies all the conditions of Lemma 3.2 and therefore f has a unique best proximity point.

Proof

Clearly, from theorem 3.2, (a) of lemma 3.2 satisfied.

To prove (b) of lemma 3.2 assume $x_n \in A$ and $y_n \in B$ are bounded sequences

Then, $d(x_n, f(x_n))$ and $d(y_n, f(y_n)) \to d(A,B)$ where x_n and y_n are sequences of A and B, respectively.

Suppose $d(x_n, f(x_n)) - d(A,B) \to 0$

Since x_n, y_n are bounded sequence, there exists subsequence n_k and $r > 0$ such that

$$d\left(f\left(x_{n_k}\right), f\left(y_{n_k}\right)\right) - d(A,B) \to r > 0$$

Clearly, $r \in X_d$.

Let $r_{n_k} = d\left(f\left(x_{n_k}\right), f\left(y_{n_k}\right)\right) - d(A,B)$ and $s_{n_k} = d\left(x_{n_k}, y_{n_k}\right) - d(A,B)$

Given $r_{n_k} - s_{n_k} \to 0$ as $k \to \infty$ from diagonal property.

$$d\left(f\left(x_{n_k}\right), f\left(y_{n_k}\right)\right) - d(A,B) \le d\left(f\left(x_{n_k}\right), f\left(y_{n_k}\right)\right) - d(A,B)$$

Therefore,

$$r_{n_k} \le \phi\left(s_{n_k}\right) \tag{33}$$

Now from 33

$$0 > \phi\left(s_{n_k}\right) - s_{n_k}$$

$$= \phi\left(s_{n_k}\right) - r_{n_k} + r_{n_k} - s_{n_k}$$

$$\ge r_{n_k} - s_{n_k}$$

Since $r_{n_k} - s_{n_k} \to 0$ we have $\lim\inf\left(\phi\left(s_{n_k}\right) - s_{n_k}\right) = 0$.

Contradicting $inf_{t \in [r_0, s]}\left(t - \phi(t)\right) > 0$ where $s_{n_k} \downarrow r_0$.

This completes the proof.

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