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Review on the Inverse Rayleigh Distribution

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ABSTRACT

This article discusses Bayesian and non-Bayesian estimation problem of the unknown parameter for the inverse Rayleigh distribution based on the lower record values. Maximum likelihood estimators of the unknown parameters were obtained. Furthermore, Bayes estimator has been developed under squared error and zero one loss functions. We discuss also statistical properties and estimation of power-transmuted inverse Rayleigh distribution (EIRD). We introduce the transmuted modified inverse Rayleigh distribution using quadratic rank transmutation map, which extends the modified inverse Rayleigh distribution. We introduce a generalization of the inverse Rayleigh distribution known as EIRD which extends a more flexible distribution for modeling life data. Some statistical properties of the EIRD are investigated, such as mode, quantiles, moments, reliability, and hazard function. We describe different methods of parametric estimations of EIRD discussed by using maximum likelihood estimators, percentile-based estimators, least squares estimators, and weighted least squares estimators and compare those estimates using extensive numerical simulations. The new two-scale parameters generalized distribution were studies with its distribution and density functions, besides that the basic properties such as survival, hazard, cumulative hazard, quantile function, skewness, and Kurtosis functions were established and derived. To estimate the model parameters, maximum likelihood, and rank set sampling estimation methods were applied with real-life data. We have introduced weighted inverse Rayleigh (WIR) distribution and investigated its different statistical properties. Expressions for the Mode and entropy have also been derived. A comprehensive account of the mathematical properties of the modified inverse Rayleigh distribution including estimation and simulation with its reliability behavior is discussed.

Key words: Bayesian estimation, Bayesian prediction transmuted inverse Rayleigh, exponential Rayleigh distribution, inverse Rayleigh distribution, lower record values, maximum likelihood, maximum likelihood, mean residual life function, modified inverse Rayleigh distribution, modified inverse Rayleigh distribution, moments, order statistics, percentiles

INTRODUCTION

The inverse Rayleigh distribution has many applications in the reliability studies. Voda (1972) mentioned that the distribution of lifetimes of several types of experimental units can be approximated by the inverse Rayleigh distribution. The probability density function of the inverse Rayleigh distribution with scale parameter θ is given by:

$$f(x)=\left(\frac{2\theta}{x^2}\right)exp\left(-\frac{\theta}{x^2}\right)x,\theta>0 \tag{1.1}$$

The corresponding cumulative distribution function (cdf) is,

$$F(x)=exp\left(-\frac{\theta}{x^2}\right)x,\theta>0 \tag{1.2}$$

Gharraph (1993) derived five measures of location for the inverse Rayleigh distribution. These measures are the mean, harmonic mean, geometric mean, mode, and the median. He also estimated the unknown parameter using different methods of estimation. A comparison of these estimators was discussed numerically in term of their bias and root mean square error (MSE). Abdel-Monem (2003) developed some estimation

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and prediction results for the inverse Rayleigh distribution. El-Helbawy and Abd-El-Monem (2005) obtained Bayesian estimators of the parameter of the inverse Rayleigh distribution under four loss functions. Bayesian of one and two sample predictions is also developed including point predictions and prediction interval.

Soliman *et al*. (2010) introduce how record values can be used to develop a methodology to construct and compute Bayesian and non-Bayesian estimation and prediction. The lower record values from inverse Rayleigh population based on a set of lower record values will be considered.

Leao *et al*. (2013) are studied the beta generalized distribution based on the IR distribution, called the beta inverse Rayleigh (BIR) distribution. The BIR distribution is a special case of the beta Fréchet (BF) distribution, The BIR probability density function (pdf) can be expressed as (for x > 0):

$$F(x) = (2\theta)/\left(B(a,b)x^3\right)exp \left(\frac{-a\theta}{x^2}\right)\left[1 - exp\left(\frac{-\theta}{\boldsymbol{x}^2}\right)\right]^{b-1} \tag{1.3}$$

The BIR random variable X is denoted by X ~ BIR (a, b, and θ). The parameters a and b affect the skewness of X by changing the relative tail weights. Simulating the BIR random variable is relatively simple. Let Y be a random variable distributed according to the usual beta distribution with parameters a and b. Thus, by means of the inverse transformation method, the random variable X is given by $\mathrm{X} = \sqrt{-\frac{\theta}{\log(y)}}$.

Khan *et al*. (2014) introduce that the modified inverse Rayleigh (MIR) distribution is the special case of the modified inverse Weibull (MIW) distribution proposed by Khan *et al*. (2012) and studied its theoretical properties. The modified inverse Rayleigh distribution approaches to the inverse Rayleigh and inverse exponential distributions when its parameters change. The modified inverse Rayleigh distribution is very useful lifetime model which can be used for analyzing lifetime data. In this research, the properties of the modified inverse Rayleigh distribution are discussed.

The cdf of the inverse Rayleigh distribution is given by:

$$G(x;\theta) = exp\left(-\theta\left(\frac{1}{x}\right)^2\right) \tag{1.4}$$

The probability density function (pdf) corresponding

$$g(x;\theta) = \frac{2\theta}{x^3} exp\left(-\theta\left(\frac{1}{x}\right)^2\right) \tag{1.5}$$

Here, θ is the scale parameter. The behavior of instantaneous failure rate of the inverse Rayleigh distribution has increasing and decreasing failure rate patterns for lifetime data

Khan *et al*. (2015) studied the modified inverse Rayleigh (MIR) distribution and discussed its theoretical properties. The cdf of the MIR distribution is given by:

$$G(x;\ \alpha,\ \beta) = exp\left(-\frac{\alpha}{x} - \beta\left(\frac{1}{x}\right)^2\right), \mathrm{x} > 0, \tag{1.6}$$

where α > 0 and β > 0 are the scale parameters. The density function corresponding to (1.6) is

$$g(x;\ \alpha,\ \beta) = \left(\alpha + \frac{2\beta}{x}\right)\left(\frac{1}{x}\right)^2 exp\left(\frac{-\alpha}{x} - \beta\left(\frac{1}{x}\right)^2\right), x > 0 \tag{1.7}$$

Fatima *et al.* (2017) introduce that the pdf of Weighted distribution of X can be defined as:

$$f(x) = \frac{x^k g(x)}{E_g\left(x^k\right)}, k \geq 0, x > 0 \tag{1.8}$$

Rao *et al*. (2019) are shown that the cumulative density function (CDF) of the exponentiated inverse Rayleigh distribution (EIRD) is given by:

$$F(x) = 1 - \left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha};\ x \geq 0, \sigma > 0, \alpha > 0 \tag{1.9}$$

where σ is the scale parameter and is the shape parameter. The probability density function (PDF) of EIRD is:

$$f(x) = \frac{2\alpha\sigma^2}{x^3} e^{-\left(\frac{\sigma}{x}\right)^2}\left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha-1};\ x \geq 0, \sigma > 0, \alpha > 0 \tag{1.10}$$

The TIR distribution is a generalization of the IR distribution using the quadratic rank transmutation map (Ahmed *et al*., 2014 and Hassan *et al*., 2020). The cdf of the TIR distribution is given by:

$$F_{TIR}(y;\theta;\lambda)=e^{-\theta y^{-2}}\left(1+\lambda-\lambda e^{-\theta y^{-2}}\right),\theta>0,\lambda\leq 1,y>0 \quad (1.11)$$

Mohammed *et al*. (2021) are shown that the random variable Y=1/X is following the inverse exponential Rayleigh distribution (IERD) with two non-negative shape parameters (γ and β), the cdf, pdf, and surveil (reliability) function are, respectively, give:

$$G(y)=e-\left(\frac{\gamma}{y}+\frac{\beta}{2y^2}\right) \quad (1.12)$$

$$g(y)=\frac{1}{y^2}\left(\gamma+\frac{\beta}{y}\right)e-\left(\frac{\gamma}{y}+\beta/y^2\right) \quad (1.13)$$

$$S(y)=1-e-\left(\gamma/y+\beta/2y^2\right) \quad (1.14)$$

ESTIMATION

Soliman *et al*. (2010) are given that the r[th] moment about origin for (m +1) lower record values from inverse Rayleigh distribution is given as follows:

$$E(r_m^k)=\frac{\theta^{\frac{k}{2}}}{m!}\text{“}\left(\frac{2m-k+2}{2}\right),k<2m+2 \quad (2.1)$$

the logarithm of the likelihood function:

$$\ln\left(L(\theta;r)\right)=(m+1)\ln(2\theta)-\frac{\theta}{r^2}-\ln\left(\prod_{i=0}^{m}r_i^3\right) \quad (2.2)$$

Differentiate both side of equation (2.2) with respect to the parameter θ and equating with zero, then the maximum likelihood estimate of θ under lower record value, say $\hat{\theta}$,

$$\hat{\theta}=(m+1)r_m^2 \quad (2.3)$$

The problem of obtaining Bayesian estimators for the scale parameter from the inverse Rayleigh distribution.

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\lambda}\end{pmatrix}\sim N\left[\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix},\begin{pmatrix}\hat{V}_{11}\hat{V}_{12}\hat{V}_{13}\\ \hat{V}_{21}\hat{V}_{22}\hat{V}_{23}\\ \hat{V}_{31}\hat{V}_{32}\hat{V}_{33}\end{pmatrix}\right],$$

the expecting information matrix is given by

$$V^{-1}=-E\begin{pmatrix}\frac{\partial^2\log L}{\partial\alpha^2} & \frac{\partial^2\log L}{\partial\alpha\partial\beta} & \frac{\partial^2\log L}{\partial\alpha\partial\lambda}\\ \frac{\partial^2\log L}{\partial\beta\partial\alpha} & \frac{\partial^2\log L}{\partial\beta^2} & \frac{\partial^2\log L}{\partial\beta\alpha\lambda}\\ \frac{\partial^2\log L}{\partial\alpha\partial\lambda} & \frac{\partial^2\log L}{\partial\lambda\partial\beta} & \frac{\partial^2\log L}{\partial\lambda^2}\end{pmatrix}.$$

The prior knowledge which is adequately represented by the natural conjugate prior distribution under two loss functions will be developed. Consider the following informative prior distribution for the scale parameter θ:

$$\pi_1(\theta)=ae^{-a\theta} \quad (2.4)$$

The posterior probability density function is obtained by combining the likelihood given in equation (2.4) and the posterior probability density $\pi_1(\theta|r)$, then;

$$\pi_1(\theta|r)=k\frac{a(2\theta)^{m+1}}{\prod_{i=0}^{m}r_i^3}exp\left[-\left(\frac{1}{r_m^2}+\alpha\right)\theta\right],\theta>0$$

where, the normalizing constant k is given by.

$$k=1/\int_\theta^\theta\pi_1(\theta|r)d\theta,A=\left[\frac{1}{r_m^2}+a\right]$$

$$\pi_1(\theta|r)=\frac{A(A\theta)^{m+1}}{\Gamma(m+2)}exp(-A\theta),\theta>0 \quad (2.5)$$

The Bayesian estimator of θ under squared error loss function is the posterior mean and is given by:

$$\ddot{\theta}_1=\frac{m+2}{A} \quad (2.6)$$

The Bayes estimator of θ with respect to zero one loss function is the posterior mode which is given by

$$\ddot{\theta}_2 = \frac{d\left(ln\pi_1\left(\theta\,|r\right)\right)}{d\theta} = \frac{m+1}{A} \quad (2.7)$$

Leao *et al*. (2013) are written the r[th] moment of X as:
$E(Xr) = \theta^{r/2}/B(a, b)\ S_r(a, b),\ r<2$ (2.8)
Where,

$$S_r\left(a,\ b\right) = \Gamma\left(\mathrm{b}\right)\Gamma\left(1-\mathrm{r}/2\right)\sum_{n=0}^{b}(-1)^n \binom{b-1}{n}\left(1/\left(a+n\right)^{1-2r}\right) \quad (2.9)$$

Khan (2014) is shown the moment and moment generating function of the distribution as:

$$\mu_k = \int_0^\infty x^{k-2}\left(\alpha + \frac{2\theta}{x}\right)exp\left(-\frac{\alpha}{x}\theta\left(\frac{1}{x}\right)^2\right)dx \quad (2.10)$$

the moment generating function of $M_x\ (t)$ is given as follows:

$$M_x\left(t\right) = \int_0^\infty \left(\alpha + \frac{2\theta}{x}\right)\left(\frac{1}{x}\right)^2 exp\left(tx - \frac{\alpha}{x}\theta\left(\frac{1}{x}\right)^2\right)dx \quad (2.11)$$

The likelihood function of (1.5) is given by:

$$L = \prod_{i=1}^{n}\left(\alpha + \frac{2\theta}{x}\right)\left(\frac{1}{x}\right)^2 exp\left(tx - \frac{\alpha}{x}\theta\left(\frac{1}{x}\right)\right)^2 \quad (2.12)$$

By taking logarithm of (2.12), we find the log-likelihood function, differentiating with respect to α, θ and then equating it to zero, we obtain the estimating equations.
Khan *et al*. (2015) are given that the k[th] moment of X has the TMIR (x; α, β, and λ) with $|\lambda| \le 1$, then is given by:

$$\mu_k = \left(1+\lambda\right)\sum_{p=0}^{\infty}\frac{\left(-1\right)^p \beta^p \alpha^{k-2p}}{p!}z_2\left(k,p\right) - \lambda\sum_{p=0}^{\infty}\frac{\left(-1\right)^p \beta^p \left(2\alpha\right)^{k-2p}}{p!}z_1\left(k,p\right), \quad (2.13)$$

$$z_g\left(k,p\right) = \left\{\Gamma\left(2p-k+1\right) + \frac{g\beta}{\alpha^2}\text{“}\left(2p-k+2\right)\right\}, g = 1.2 \quad (2.14)$$

Consider the random samples $x_1, x_2, \ldots, x_n$ consisting of n observations from the TMIR distribution and $\Theta = (\alpha, \beta, \lambda)$ T be the parameter vector. The log likelihood function of (1.7) is given by:
It is more convenient to use quasi-Newton algorithm to numerically maximize the log-likelihood function given in above equation to yield the ML estimators $\hat{\alpha}, \hat{\beta}$, and $\hat{\lambda}$, respectively. For finding the interval estimation and testing the hypothesis of the subject model.

$$\hat{\alpha} \pm Z\frac{\alpha}{2}\sqrt{\hat{V}_{11}}, \hat{\beta} \pm Z\frac{\alpha}{2}\sqrt{\hat{V}_{22}}, \hat{\lambda} \pm Z\frac{\alpha}{2}\sqrt{\hat{V}_{33}},$$

Fatima *et al*. (2017) are shown that the r[th] moment of a continuous random variable X is given as follow:

$$\mu_r = E(X^r) = \frac{\lambda^{r/2}}{\Gamma\left(1-\frac{k}{2}\right)}\Gamma\left(1-\frac{r+k}{2}\right) \quad (2.15)$$

Then moment generating function of X denoted by $M_x\ (t)$ is given by:

$$M_x\left(t\right) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{\lambda^{r/2}}{\Gamma\left(1-k/2\right)}\Gamma\left(1-\frac{r+k}{2}\right) \quad (2.16)$$

We make use of the method of maximum likelihood estimation (MLE):

$$L(x) = \frac{2^n \lambda^{n(1-\frac{k}{2})}}{\Gamma\left(1-k/2\right)}\prod_{i=1}^{n}x_i^{k-3}e^{\frac{-\lambda}{x_i^2}} \quad (2.17)$$

By taking logarithm and differentiating equation and equate to zero, we get $\hat{\lambda}$.
Rao *et al.* (2019) are shown moments and moment generating function. The r[th] moment about origin is given by:

$$\mathrm{Log\,L} = \sum_{I=1}^{n}\log\left(\alpha + \frac{2\beta}{x_i}\right) + \sum_{I=1}^{n}\log\left(\frac{1}{x_i}\right)^2 - \sum_{I=1}^{n}\left(\frac{\alpha}{x_i}\right) - \beta\sum_{I=1}^{n}\left(\frac{\alpha}{x_i}\right)^2 + \sum_{I=1}^{n}\log\left\{1+\lambda-2\lambda\exp\left(-\frac{\alpha}{x_i}-\beta\left(\frac{\alpha}{x_i}\right)^2\right)\right\} \quad (2.18)$$

$$\mu_r^{`} = \alpha \sum_{j=0}^{\infty} \binom{\alpha - 1}{j} (-1)^j \frac{(\sigma^2 (j+1)^{r/2}}{j+1} \text{``} \left(1 - \frac{r}{2}\right)$$

(2.19)

The moment generating function is given by:

$$M_x(t) = \alpha \sum_{r=0}^{\infty} \binom{\alpha - 1}{r} \frac{(-t)^r}{r!} \frac{(\sigma^2 (r+1)^{r/2}}{r+1} \tilde{A} \left(1 - \frac{r}{2}\right)$$

(2.20)

MLE method is mostly used in many writings. The MLE satisfies many properties of good estimator.

$$L = 2^n \sigma^{2n} \alpha^n \prod_{i=1}^{n} x_i^{-3} e^{-\sum_{i=1}^{n} \left(\frac{\sigma}{x_i}\right)^2} \prod_{i=1}^{n} \left(1 - e^{-(\frac{\sigma^2}{x_i})}\right)^{\alpha - 1}$$

(2.21)

Equation above can be solved by iterative procedure like Newton Raphson method to obtain the ML estimator.

Hassan *et al.* (2020) are shown the moments under various characteristics of a frequency distribution. They have been applied to obtain mean and variance, in addition to some measures, such as skewness and kurtosis. The r^{th} moment of X has the PTIR distribution and is derived using (1.3) as follows:

$$E(X^r) = \int_0^{\infty} x^r \left[\frac{2\theta\beta}{x^{2\beta+1}} e^{\frac{-\theta}{x^{2\beta}}} \left(1 + \lambda - 2e^{\frac{-\theta}{x^{2\beta}}} \right) \right] dx$$

(2.22)

$$E(X^r) = \theta^{\frac{x}{2\beta}} \Gamma\left(1 - \frac{r}{2\beta}\right) \left[1 + \lambda - 2^{\frac{r}{2\beta}} \lambda\right],$$

$r < 2\beta, r = 1.2.3, \ldots$ (2.23)

The ML estimator procedure is considered to estimate the population parameters of the PTIR distribution. The likelihood function is given by:

$$L = (2\theta\beta) \prod_{i=1}^{n} x_i^{-(1+2\beta)} e^{\frac{-\theta}{x^{2\beta}}} (1 + \lambda - 2\lambda e^{\frac{-\theta}{x^{2\beta}}} \quad (2.24)$$

Then ML estimators of the parameters θλ and β denoted by $\hat{\theta}, \hat{\lambda}$ and $\hat{\beta}$ are determined by solving numerically the non-linear equations for first differentiation after setting them equal to zeros simultaneously.

Mohammed *et al.* (2021) are shown the general form of r^{th} non-central moment. In the case of finding the moment of IERD and because of the complexity involved in the integration form, specialist mathematics models are used such as Tylor's series expansion and Gamma function.

$$M_r^{`} = \int_0^{\infty} y^{r-2} \left(\gamma + \frac{\beta}{y}\right) e^{-\left(\frac{\gamma}{y} + \frac{\beta}{2y^2}\right)} dy \quad (2.25)$$

MLE method is given by:

$$L_{MLE}(y_1, y_2, \ldots, y_n; \gamma, \beta) \prod_{i=1}^{n} \left(\frac{1}{y_i^2}\right) \left(\gamma + \frac{\beta}{y_i}\right) e^{-\left(\frac{\gamma}{y_i} + \frac{\beta}{2y_i^2}\right)}$$

(2.26)

To obtain the results that represent the parameters estimated by the MLE method, numerical methods are obtained.

SIMULATION STUDY

In this section, a simulation study is presented to illustrate the application of the various theoretical results developed in the previous sections.

Abdel-Monem (2003) is used the cdf as the method of simulation studies. Using Mathematica 9 programing to obtain some tables that summarize the results of a simulation study from sample size n = 20, 30, 40, and 50 from the IRD with $\theta = 5$, 10, and 15.

The mle $\hat{\theta}$ and the estimators $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5$ present the generalized samples of sizes n = 20, 30, 40, and 50 from the IRD. It estimates decrease as the sample sizes increase for selected sets of parameters.

El-Helbawy *et al.* (2005) obtained Bayesian estimators of the parameter of the inverse Rayleigh distribution under four loss functions. Bayesian of one and two sample predictions is also developed including point predictions and prediction interval using Matlab 7. It is seen from tables that the estimators of θ (for = 0.5 and 2) as increases and the results n increases for complete samples are

Table 1: The Bayesian estimators of θ under three loss functions using complete samples.

n	n_s	Mean of the estimator and v $(\hat{\theta})$ under the squared error loss function				Mean of the estimator and v$(\hat{\theta})$ under the zero-one loss function				Mean of the estimator and v$(\hat{\theta})$ under the LINEX loss function			
		M $(\hat{\theta}_1)$	V $(\hat{\theta}_1)$	M $(\hat{\theta}_2)$	V $(\hat{\theta}_2)$	M $(\hat{\theta}_1)$	V $(\hat{\theta}_1)$	M $(\hat{\theta}_2)$	V $(\hat{\theta}_2)$	M $(\hat{\theta}_1)$	V $(\hat{\theta}_1)$	M $(\hat{\theta}_2)$	V $(\hat{\theta}_2)$
50	100	0.5121	0.0046	0.0484	0.0733	0.5019	0.0044	2.0074	0.0704	0.5095	0.0045	2.0069	0.0676
	500	0.5112	0.0058	2.0449	0.0932	0.5010	0.0056	2.0040	0.0895	0.5086	0.0057	2.0034	0.0858
	1000	0.5089	0.0052	2.0355	0.0825	0.4987	0.0050	1.9948	0.0792	0.5063	0.0051	1.9944	0.0760
	5000	0.5109	0.0053	2.0436	0.0840	0.5007	0.0050	2.0027	0.0807	0.5083	0.0051	2.0022	0.0773
100	100	0.5013	0.0023	2.0052	0.0372	0.4963	0.0023	1.9852	0.0365	0.5000	0.0023	1.9852	0.0357
	500	0.5059	0.0024	2.0238	0.0390	0.5009	0.0024	2.0035	0.0382	0.5047	0.0024	2.0034	0.0374
	1000	0.5025	0.0024	2.0099	0.0386	0.4975	0.0024	1.9898	0.0379	0.5012	0.0024	1.9898	0.0371
	5000	0.5050	0.0026	2.0200	0.0409	0.5000	0.0025	1.9998	0.0401	0.5037	0.0025	1.9997	0.0393
200	100	0.5032	0.0015	2.0126	0.0247	0.5006	0.0015	2.0026	0.0245	0.5025	0.0015	2.0025	0.0242
	500	0.5026	0.0013	2.0106	0.0121	0.5001	0.0013	2.0005	0.0209	0.5020	0.0013	2.005	0.0207
	1000	0.5028	0.0013	2.0114	0.0203	0.5003	0.0013	2.0013	0.0201	0.5022	0.0013	2.0013	0.0199
	5000	0.5026	0.0013	2.0105	0.0207	0.5001	0.0013	2.0005	0.0205	0.5020	0.0013	2.004	0.0203
300	100	0.5001	8.7558e-004	2.0006	0.0140	0.4985	8.6975e-004	1.9939	0.0139	0.4997	8.7262e-004	1.9939	0.0138
	500	0.5026	8.7771e-004	2.0105	0.0110	0.5010	8.7187e-004	2.0038	0.0139	0.5022	8.7477e-004	2.0038	0.0139
	1000	0.5019	8.3502e-004	2.0076	0.0134	0.5002	8.2946e-004	2.0009	0.0133	0.5015	8.7222e-004	2.0009	0.0132
	5000	0.5012	7.9010e-004	2.0049	0.0126	0.4996	7.8484e-004	1.9983	0.0126	0.5008	7.87464-001	1.9983	0.0125
400	100	0.5022	4.8488e-004	2.0086	0.0078	0.5009	4.8246e-004	2.0036	0.0077	0.5018	4.8366e-004	2.0036	0.0077
	500	0.5007	5.7110e-004	2.0028	0.0092	0.4994	5.7422e-004	1.9978	0.0092	0.5004	5.7565e-004	1.9978	0.0091
	1000	0.5008	5.6495e-004	2.0031	0.0090	0.4995	5.6212e-004	1.9981	0.0090	0.5005	5.6353e-004	1.9981	0.0089
	5000	0.5018	6.4561e-004	2.0072	0.0103	0.5005	6.4239e-004	2.0022	0.0103	0.5015	6.4399e-004	2.0022	0.0102
500	100	0.5023	5.5978e-004	2.0093	0.0090	0.5013	5.5755e-004	2.0053	0.0089	0.5021	5.5866e-004	2.0063	0.0089
	500	0.5014	4.8046e-004	2.0055	0.0077	0.5004	4.7854e-004	2.0015	0.0077	0.5011	4.7949e-004	2.0015	0.0076
	1000	0.5011	5.0161e-004	2.0045	0.0080	0.5001	4.9960e-004	2.0005	0.0080	0.5009	5.0060e-004	2.0005	0.0080
	5000	0.5008	5.1174e-004	2.0031	0.0082	0.4998	5.0970e-004	1.9991	0.0082	0.5005	5.1071e-004	1.9991	0.0081

slightly better than the corresponding results for Type II censored samples.[1-10]

Soliman *et al*. (2010) are shown that numerical results of the Bayesian predictive interval for several values of different prior parameters will be obtained. The calculations are carried out according to the following steps: (1) For given values of the inverse Rayleigh parameters θ generate a random variable X from the inverse Rayleigh distribution (1.1) and selected the first 10 records. (2) Consider the first six records as the observed upper records (m=5), while the last six records as the unobserved records, which are to be predicted. Using Mathcad (2001) program to obtain the 95% equal tail Bayesian prediction interval for the s^{th} upper record values for m = 5 and s = 6 and for several different values of the prior parameters (a = 0.5, 1, 2, 3, 5, and 7) and Tables 1 and 2 contained the results which show (i) the values of the prior parameters a, (ii) the 95% Bayesian prediction interval for the s^{th} records, and (iii) the lengths of the prediction interval.

Liao *et al*. (2013) are studied the beta inverse Rayleigh distribution as a generalization of the inverse Rayleigh distribution. They are provided a better foundation for some mathematical properties for this distribution, including the derivation of the hazard rate function, moments, quantile measures, mean deviations, entropy measures, and order statistics. The model parameters are estimated by maximum likelihood. An application of the BIR distribution to a real data set indicates that this distribution outperforms both the exponentiated inverse Rayleigh and inverse Rayleigh distributions.

Khan (2014) introduced the MIR distribution, which is an extension of the IR distribution. The new parameter provides more flexibility in modeling reliability data. Some of its properties are discussed illustrating the usefulness of the MIR distribution to real data using MLE. The likelihood ratio test concludes that the MIR distribution provides consistent result than the IR and IE distributions.

Khan *et al*. (2015) proposed a new distribution, named the TMIR distribution, which is an extension of the MIR distribution. The TMIR distribution provides better results than the MIR,

Table 2: The mean of the Baysian estimator of θ under three loss function using a type II censored sample with m=0.98(n)

n	m	n_s	Mean of the estimator and v($\hat{\theta}$) under the squared error loss function				Mean of the estimator and v($\hat{\theta}$) under the zero-one loss function				Mean of the estimator and v($\hat{\theta}$) under the LINEX loss function			
			M ($\hat{\theta}_1$)	V ($\hat{\theta}_1$)	M ($\hat{\theta}_2$)	V ($\hat{\theta}_2$)	M ($\hat{\theta}_1$)	V ($\hat{\theta}_1$)	M ($\hat{\theta}_2$)	V ($\hat{\theta}_2$)	M ($\hat{\theta}_1$)	V ($\hat{\theta}_1$)	M ($\hat{\theta}_2$)	V ($\hat{\theta}_2$)
50	49	100	0.5017	0.0044	2.0067	0.0704	0.4914	0.0042	1.9657	0.0675	0.4991	0.0043	0.9600	0.649
		500	0.5033	0.0052	2.0132	0.0830	0.4930	0.0050	1.9721	0.0796	0.5007	0.0051	1.9721	0.0763
		1000	0.5005	0.0049	2.0022	0.0777	0.4903	0.0047	1.9613	0.0746	0.4980	0.0048	1.9616	0.0716
		5000	0.4989	0.0049	1.9955	0.0788	0.1887	0.0047	1.9548	0.0756	0.4963	0.0048	1.9552	0.0726
100	98	100	0.5037	0.0029	2.0146	0.0465	0.4985	0.0028	1.9941	0.0455	0.5024	0.0029	1.9940	0.0446
		500	0.4987	0.0028	1.9946	0.0444	0.4936	0.0027	1.9743	0.0435	0.4974	0.0027	1.9744	0.0426
		1000	0.4926	0.0023	1.9703	0.0371	0.4876	0.0023	1.9502	0.0364	0.4913	0.0023	1.9506	0.0357
		5000	0.4944	0.0025	1.9777	0.0407	0.4849	0.0025	1.9575	0.0399	0.4932	0.0025	1.9578	0.0391
200	196	100	0.4889	0.0012	1.9558	0.0191	0.4865	0.0012	1.9458	0.0189	0.4883	0.0012	1.9461	0.0187
		500	0.4932	0.0011	1.9729	0.0183	0.4907	0.0011	1.9629	0.0181	0.4926	0.0011	1.9630	0.0179
		1000	0.4919	0.0012	1.9674	0.0187	0.4894	0.0012	1.9574	0.0185	0.4912	0.0012	1.9576	0.0183
		5000	0.4931	0.0013	1.9724	0.0202	0.4906	0.0012	1.9623	0.0200	0.4925	0.0013	1.9625	0.0198
300	294	100	0.4932	7.5342e-004	1.9729	0.0121	0.4916	7.4830e-004	1.9662	0.0120	0.4928	7.5089e-004	1.9663	0.0119
		500	0.4930	8.5151e-004	1.9719	0.0136	0.4913	8.4572e-004	1.9652	0.0135	0.4926	8.4864e-004	1.9653	0.0134
		1000	0.4932	8.0078e-004	1.9730	0.0128	0.4916	7.9535e-004	1.9663	0.0127	0.4928	7.9809e-004	1.9664	0.0126
		5000	0.4922	7.8490e-004	1.9688	0.0126	0.4905	7.7957e-004	1.9621	0.0125	0.4918	7.8226e-004	1.9622	0.0124
400	392	100	0.4885	6.3058e-004	1.9539	0.0100	0.4872	6.2736e-004	1.9490	0.0100	0.4882	6.2901e-004	1.9491	0.0100
		500	0.4924	5.8017e-004	1.9697	0.0093	0.4912	5.7721e-004	1.9647	0.0092	0.4921	5.7871e-004	1.9647	0.0092
		1000	0.4899	5.9112e-004	1.9596	0.0095	0.4887	5.8810e-004	1.9546	0.0049	0.4896	5.8963e-004	1.9547	0.0049
		5000	0.4906	6.1077e-004	1.9624	0.0098	0.4894	6.0766e-004	1.9574	0.0097	0.4903	6.0929e-004	1.9575	0.0097
500	490	100	0.4902	4.8571e-004	1.9608	0.0078	0.4892	4.8373e-004	1.9568	0.0077	0.4900	4.8473e-004	1.9569	0.0077
		500	0.4937	4.9643e-004	1.9747	0.0079	0.4927	4.9441e-004	1.9707	0.0079	0.4934	4.9543e-004	1.9707	0.0079
		1000	0.4890	5.0147e-004	1.9558	0.0080	0.4880	4.9943e-004	1.9518	0.0080	0.4887	5.0047e-004	1.9519	0.0080
		5000	0.4909	4.8196e-004	1.9637	0.0077	0.4899	4.7999e-004	1.9597	0.0077	0.4907	4.8099e-004	1.9598	0.0076

TIR, and IR distributions. In this model, the new parameter λ provides more flexibility in modeling reliability data. They derived the quantile function, moments, moment generating function, entropies, mean deviation, Bonferroni, and Lorenz curves. They derived the S^{th} moment of r^{th} order statistics and the k^{th} moment of r^{th} median order statistics. They discussed the MLE and obtained the fisher information matrix. The usefulness of the new model is illustrated in an application to real data using MLE. They hoped that the proposed model may attract wider application in the analysis of reliability data.

Fatima *et al*. (2017) introduced weighted inverse Rayleigh (WIR) distribution, which acts as a generalization to so many distributions, namely, IRD, LBIRD, WRD, LBRD, and RD. After introducing WIRD, we investigated its different mathematical properties. Two real data sets have been considered to make comparison between special cases of WIRD in terms of fitting. After the fitting of WIRD and its special cases to the data sets considered, WIRD possesses minimum values of Akaike information criterion (AIC), corrected AIC, and Bayesian information criterion on its fitting, to two real life data sets. Therefore, we can conclude that the WIRD will be treated as a best fitted distribution to the data sets as compared to its other special cases.

Rao *et al*. (2019) proposed Monte Carlo simulation study. They are conducted to evaluate the performance of different simulation method for estimating unknown parameters of EIRD. The performance of the different estimators is evaluated in terms of MSE. The simulation is conducted using R-software, 10,000 random samples of EIRD were generated with values of n = (20, 40, 50, and 100) while choosing (α, σ) = (0.5, 1), (1.5, 1), (2, 1), (2.5, 1), (0.5, 2), (1.5, 2), (2, 2), and (2.5, 2). Average bias and MSE values obtained by the method of MLE, LSE, WLSE, and PCE. EIRD which is derived from this study performs well; from the diagram of the PDF, it can be shown that the distribution is positively skewed, and the CDF shows the increasing pattern as other distributions. Furthermore, using reliability function, the distribution can be used in lifetime studies since reliability graph tends to decrease as the time increases. The hazard function shows the upside-down bath-tub curve shape. The unique characteristic of the distribution has only one moment, and kurtosis and skewness are found in terms of quantile. Four methods of estimation were used in parameter estimation; the methods are maximum likelihood, least square, weighted least square, and percentile estimation. From the simulation study, it is observed that the method of maximum likelihood is the best compared to other methods since it has minimum value of MSE. Furthermore, findings revile that all methods are consistent since the values of bias and MSE decrease as sample size increases. The data set of coating weights for March 2018 from ALAF industry is used to study the performance of the proposed distribution. It is shown that EIRD is better performed more than the existing distributions, namely: IRD, RD, IWD, and GIED.

Hassan *et al*. (2020) is performed a numerical study to evaluate and compare the performance of the estimates with respect to their absolute biases (ABs) and MSEs for different sample sizes and for different parameter values. The numerical procedures are described as follows: Step (1): A random sample $X_1,\ldots$, Xn of sizes n = 10, 20, 30, and 100 is selected. These random samples are generated from the PTIR distribution.

Step (2): Four different set values of the parameters are selected as: Set 1 = ($\theta = 1.0$, $\lambda = 0.5$, and $\beta = 0.5$), Set 2 = ($\theta = 1.0$, $\lambda = 0.5$, and $\beta = 1.5$), Set 3 = ($\theta = 1.0$, $\lambda = 0.5$, and $\beta = 2$), and Set 4 = ($\theta = 0.5$, $\lambda = -0.7$, and $\beta = 1$). Step (3): The ML, LS, and PR estimates of θ, λ, and β are computed for each set of parameters and for each sample size. Step (4): Steps from 1 to 3 are repeated 5000 times for each sample size and for selected sets of parameters. Then, the ABs and MSEs of the ML, LS, and PR estimates are computed. The following conclusions can be observed on the properties of estimated parameters. The MSEs of the ML, LS, and PR estimates decrease as the sample sizes increase for selected sets of parameters. The MSEs for the ML estimates of θλ and β take the smallest values compared to the MSEs of the LS and PR estimates in almost all of the cases. The ABs of the ML estimates are smaller than the ABs of the PR and LS estimates in almost all of the cases especially at small and moderate sample sizes. The ABs and MSEs of the ML, PR, and LS estimates of β are smaller than the corresponding estimates of θ and λ in almost all of the cases.

Mohammed *et al*., 2021 adopted and studied a new approach to mixing the distributions in addition to the inverse distribution approach. The new two-parameter lifetime distribution is called IERD. The statistical properties such as probability density, cumulative, survival, hazard, quantile, and moment functions were provided by this study. The new model parameters were estimated using maximum likelihood and ranked set sampling methods with simulation studies of different sizes to show the general behavior of the new model in terms of flexibility and effectiveness.[11-15]

CONCLUSION

We proposed a new distribution, named the TMIR distribution, which is an extension of the MIR distribution. The TMIR distribution provides better results than the MIR, TIR and IR distributions. In this model the new parameter λ provides more flexibility in modeling reliability data. We derive the quantile function, moments, moment generating function, entropies, mean deviation, Bonferroni and Lorenz curves. We also derive the Sth moment of rth order statistics and the kth moment of rth median order statistics. We discuss the maximum likelihood estimation and obtain the fisher information matrix. The usefulness of the new model is illustrated in an application to real data using MLE. We hope that the proposed model may attract wider application in the analysis of reliability data.

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