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RESEARCH ARTICLE

Simulation of the Movement of Ground Water in a rectangular jumper with a screen

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ABSTRACT

Within the frame work of planar steady-state filtration of incompressible fluid according to Darcy's law, an exact analytical solution of the problem of flow in a rectangular cofferdam with a screen in the presence of evaporation from the free surface of groundwater is given. The limiting cases of the considered motion – filtration in unconfined reservoir to imperfect gallery, as well as the flow in the absence of evaporation – are noted.

Key words: Ground water, Jumper, Evaporation, Filtration

INTRODUCTION

The solution of the problem of fluid inflow to an imperfect well with aflooded filter (i.e., axisymmetric problem) in the exact hydrodynamic formulation is associated with great mathematical difficulties (especially for flows with a free surface) and is not available so far[1-6] (numerous numerical and approximate solutions are not considered here). Therefore, as a first approximation to the solution of this problem, its flat analogues – problems about fluid flow to a rectangular cofferdam with a screen and to an imperfect rectilinear gallery – were considered,[1,5-8] which give a certain qualitative insight into the possible dependence of filtration characteristics on the degree of well imperfection. Exact analytical solution of the problem of groundwater movement in unconfined reservoir to imperfect gallery in presence of evaporation from free surface is given in work.[9] As well as an approximate solution of the problem in the case, when the flow area on the left is limited by some equipotential defined from the solution. It is shown that the flow pattern near the impermeable screen significantly depends not on lyon the imperfection of the gallery, but also on the presence of evaporation, which strongly affects the flow rate of the gallery and the ordinate of the exit point of the depression curve on the impermeable wall.

The presented work gives an exact solution of the filtration problem in a rectangular cofferdam with a screen in the presence of evaporation from the free surface of ground water. In this case, as well as in[9] (unlike in[7,8]) in the area of the flow velocity, hodograph appears not rectilinear, but circular polygons, which does not allow using the classical Christoffel-Schwarz formula. The effect of evaporation from the free surface is studied using the method of P.Y. Polubarinova-Kochina.[1-6] Using the methods of conformal mapping of circular polygons developed for special form regions,[10-13] the mixed multi parameter boundary value problem of the theory of analytic functions is solved. Taking in to account, the characteristic features of the flow under consideration makes it possible toobta in the solution through helementary functions, which makes their use the simplest and most convenient. The results of numerical calculations are given and hydrodynamic analysis of the in fluenceo fall physical parameters of the model on filtration characteristics is given. Obtained results of plane problem solution give at least some qualitative insight into dependence of flow parameters on degree of well (ortube well) imperfection.

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MATERIALS AND METHODS

Figure 1 shows a rectangular jumper with slopes A_0A_1 and D_0B_0 nanim permeable horizontal bed of length L. The height of water in the upstream reservoir is H, the downstream reservoir with water level H_1, having a partially impermeable vertical wall CD_0 (screen), is adjacent to the bottom of the reservoir. If the working part of the cofferdam CB (filter) of width H_1 is flooded, that is, $H_2>H_1$, there is no draw down gapusual for dams.[1] The upper boundary of the region of motion is the free surface A_D overlooking the impermeable screen CD_0, from which there is uniform evaporation of intensity ε (0 < ε <1). The ground is considered homogeneous and isotropic; the fluid flow obeys the Darcy law with known filtration coefficient k = const.

Let us introduce a complex potential of motion $\omega = \varphi + i\psi$, where the velocity potential, the current function and the complex coordinate are referred to kH, where H_1 is the head at point A. At the choice of the coordinate system indicated in Figure 1 and coincidence of the head comparison plane with the plane $y = 0$ at the boundary of the filtration region, the following boundary conditions are fulfilled:

$$AD: \phi = -y,\ \psi = -\varepsilon x + Q;\ DC: x = 0,\ \psi = Q;$$
$$CB: x = 0,\ \phi = -H2;\ BA1: y = 0,\ \psi = 0;$$
$$A1A: \phi = -H,\ x=-L. \qquad (1)$$

The task is to determine the position of the free surface AD and to find the ordinate H_0 of the exit point of the depression curve on the screen, as well as the filtration flow rate ω.

To solve the problem, we use the method of P.Y. Polubarinova-Kochina, which is based on the application of the analytic theory of the Fuchs class linear differential equations.[1,6,14]. We introduce an auxiliary canonical variable ζ and functions: z (ζ) conformally mapping the upper half-plane ζ>0 to the flow region z under the correspondence of points $\zeta_D = 0$, $\zeta_E = e$, $\zeta_A = 1$, $\zeta_{A1} = a_1$, $\zeta_B = b$ (a_1, b are unknown affixes of points A_1 and B in the plane ζ), $\zeta_C = \infty$, and functions $d\omega/d\zeta$ and $dz/d\zeta$. We emphasize that, compared to,[9] an additional boundary angular singular point A_1 appears here in the flow region z, which complicates the solution considerably.

By determining the characteristic indices of the functions $d\omega/d\zeta$ and $dz/d\zeta$ near regular singular points,[1,6,14] we find that they are linear combinations of two branches of the following R iemann function: [1,6,14]

$$\left\{\begin{matrix} 0 & 0 & 1 & \zeta_A & \zeta_B & \infty & \\ 0 & 0 & \frac{1+\nu}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & \zeta \\ -\frac{1}{2} & 2 & -\frac{1-\nu}{2} & 1 & 1 & 2 & \end{matrix}\right\} = \frac{Y}{\sqrt{\zeta(1-\zeta)^{1+\nu}(\zeta_A-\zeta)(\zeta_B-\zeta)}},$$

$$Y = P\left\{\begin{matrix} 0 & e & 1 & \infty \\ 0 & 0 & 0 & -\frac{1+\nu}{2} \\ 1 & 2 & \nu & \nu \end{matrix}\right\} \qquad (2)$$

Where $\nu\pi = 2\mathrm{arctg}\sqrt{\varepsilon}$. The last Riemann symbol corresponds to the following Fuchs class linear differential equation with four regular singular points:

$$Y'' + \left(\frac{1}{2\zeta} + \frac{1-\nu}{\zeta-1} - \frac{1}{\zeta-e}\right)Y' + \frac{\nu(1+\nu)\zeta + \lambda}{4\zeta(\zeta-1)(\zeta-e)}Y = 0. \qquad (3)$$

It is well known[1-6,14] that difficulties of principle character arise during integration of equations of this kind. They are caused by the fact that the coefficients of equation (3) besides the uncertain affixes also contain an additional, so called accessory parameter λ, also unknown beforehand, and so far there is no effective way of their actual finding.

Let us turn to the region of the complex velocity w corresponding to the boundary conditions (1), which is depicted in Figure 2. This region, which is a circular quadrilateral $ABCDE$ with a cut with a vertex at point E (corresponding to the inflection point of the depression curve) and an

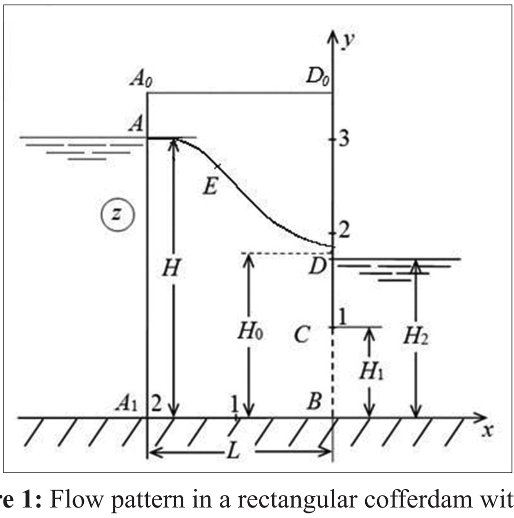

Figure 1: Flow pattern in a rectangular cofferdam with a screen, calculated at ε =0.5, H=3, L =2, H1=1, $H2$=1.4

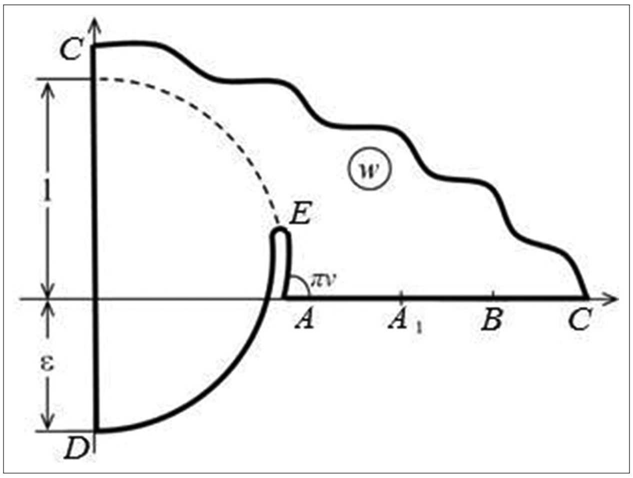

Figure 2: Area of complex velocity w

angle *vπ* at thevertex *A*, belongs to the class of circular polygons in polar meshes and was studied earlier.[13] It is important to emphasize that such areas, despite their particular form, however, are very typical and typical for many problems of underground hydromechanics: Infiltration from canals, irrigator sand reservoirs, in fresh water currents over resting saline waters, and in problems of Zhukovsky sheet flowing in the presence of saline retaining waters (for example,[11-15]).

Replacing the variables ζ = th^2t translates the upper half-plane ζ into the horizontal half-plane Re*t >0*, 0< Im *t*<0.5π of the parametric plane *t* in agreement with the points

$$t_A = \infty,\ t_D = 0,\ t_C = 0{,}5\pi, t_B = \text{arcth}\sqrt{b} + 0.5\pi, t_{A_1} = \text{arcth}\sqrt{a_1}\,(1 < a_1 < b < \infty),$$

and the *Y* integrals of equation (3), which were constructed by the method,[13] transforms to the form

$$Y_1 = \frac{\text{ch}t\text{ch}vt + C\text{sh}t\text{ch}vt}{\text{ch}^{1+v}t}, Y_2 = \frac{\text{ch}t\text{sh}vt + C\text{sh}t\text{ch}vt}{\text{ch}^{1+v}t}, \quad (4)$$

Where *C* (*C≠1*) is an unknown fitting constant.

Taking in to account relation (2) and considering that *w= d* ω/*d* ζ, we arrive at the dependencies we are looking for

$$\frac{d\omega}{t} = iMdt\frac{\sqrt{\varepsilon}\,(\text{ch}t\text{chn}t + C\text{sh}t\text{shn}t) + i(\text{ch}t\text{shn}t + C\text{sh}t\text{chn}t),}{\Delta(t)}$$

$$\frac{dz =}{dt} - \frac{M\text{ch}t\text{chn}t + C\text{sh}t\text{shn}t - i}{\sqrt{\varepsilon}}$$

$$\frac{M\text{ch}t\text{chn}t + C\text{sh}t\text{shn}t - i}{\Delta(t)}$$

$$\Delta(t) = \sqrt{[(a-1)_1\text{sh}^2t + a][(b-1)_1\text{sh}^2t + b} \quad (5)$$

Where *M > 0 is* the scale constant of the simulation. One can check that the functions (5) satisfy the boundary conditions (1) reformulated in terms of the functions *dω/dt* and *dz/dt* and, thus, are the parametric solution of the original boundary value problem. Writing representations (5) for different parts of the half-belt boundary followed by integration over the whole contour of the area of the parametric variable *t* leads to the closure of the flow area and, thus, serves to control the calculations.

RESULTS

As a result, we get expressions for the following values: the width *L* of the cofferdam, the water levels in the upper *H* and lower H_2 pools, and the length H_1 of the filter

$$\int_0^\infty X_{DA}(t)dt = L, \int_{\sqrt{\text{arcth}a1}}^\infty YAA_1(t)dt = H,$$

$$\int_0^{0.5p} [\Phi_{DC}(t) + Y_{DC}(t)]dt + H_1 = H_2$$

$$\int_0^{\sqrt{\text{arcth}b}} YCB(t)dt = H1, \quad (6)$$

Of the required coordinates of the free surface points *AD*

$$x(t) = -\int_0^\infty X_{DA}(t)dt, y(t) = H0 - \int_0^t Y_{DA}(t)dt \quad (7)$$

And expressions for the filtration flow rate *Q* and the free surface exit point ordinate

$$Q = \int_0^{\text{arcth}\sqrt{b}} \Psi_{CB}(t)dt, H0 = H - \int_0^\infty \Phi_{DA}(t)dt \quad (8)$$

Other expressions for *Q, H0* and *L* are used to control the calculations

$$Q = -\text{e}L + \int_{\text{arcth}\sqrt{a1}}^\infty \Psi_{AA}(t)\,dt, H_0 = H_2$$

$$-\int_0^{0.5p} \Phi_{DC}(t)\,dt, H_0 = H_1 +$$

$$\int_0^{0.5p} \Upsilon_{DC}(t)\,dt, L = \int_{\text{arcth}\sqrt{b}}^{\text{arcth}\sqrt{a1}} X_{BA}(t)\,dt, \quad (9)$$

As well as the expression

$$\int_0^{\infty}\Phi_{DA}(t)dt-\int_0^{0.5\pi}\Phi_{DC}(t)dt-\int_{\mathrm{arcth}b\sqrt{}}^{\mathrm{arcth}a\sqrt{}}\Phi_{BA}(t)dt. \quad (10)$$

Directly derived from the boundary conditions (1). In formula (5)-(10), the integrand functions are expressions of the right-hand sides of equations (3) on the corresponding parts of the contour of the auxiliary region *t*.

Limit cases

1. At $L\to\infty$, that is, at the junction of points A_1 and A, in plane *t*, that is, at $a_1\to 1$ (arcth $a_1=\infty$), the cofferdam degenerates in to a semi-in finite left-handed unconfined formation. Thus, the exact solution of groundwater flow to the imperfect gallery, studied earlier,[9] is obtained.
2. At $\varepsilon\to 0$, that is, at small values of evaporation intensity the results of works[7,8] are obtained.

Representations (5) - (10) contain four unknown constants *M*, *C*, *a*1 and *b*. The parameters *a*1, *b* ($1<a1<b<\infty$), *C* ($C\neq 1$) are determined from equations (6) for the given values *H1*, *H2* ($H1\leq H2<H$), and *L*, while the simulation constant *M* is found from the second equation (6) fixing the water level *H* in the headwater of the cofferdam. After determination of the unknown constants the filtration flow rate *Q* and the ordinate H_0 of the outlet point of the depression curve on the impermeable section *DC* by formulas (8) and coordinates of points of free surface *DA* by formulas (7) are sequentially found.

Figure 1 shows the flow pattern calculated at $\varepsilon = 0.5$, $H = 3$, $L = 2$, $H_1= 1.0$, $H_2= 1.4$ (basecase[9]). Results of calculations of influence of defining physical parameters ε, *H*, H_1, H_2, and *L* on values *Q* and H_0 are presented in Tables 1 and 2. On Figures 3 and 4 dependences of flow *Q* (curves 1) and ordinate H_0 of a point of an exit of a depression curve on a screen (curves 2) from parameters H_1 and H_2 are submitted. Analysis of calculations of the se tables and graphs allows us to draw the following conclusions:

- Decrease of intensity of evaporation ε and increase of head *H* accompany increase of flow *Q* and ordinate H_0 of the exit point of depression curve on the screen
- Decrease of the screen depth h_1 and increase of the water level in the downstream h_2 are accompanied by a decrease of the flow *Q* and increase of the ordinate H_0

Table 1: Results of calculations of *Q* and H_0 values when varying ε, *H* and *L*

ε	Q	H_0	H	Q	H_0	L	Q	H_0
0.1	1.3937	2.3003	2.5	0.5624	1.4074	1.5	1.6261	2.1424
0.2	1.3423	2.1544	3.0	1.1554	1.7750	1.7	1.3492	1.8970
0.3	1.2839	2.0179	3.5	1.5715	2.0883	2.0	1.1554	1.7755
0.4	1.2218	1.8920	4.5	2.6811	3.3097	2.5	0.7585	1.5045
0.5	1.1554	1.7755	5.0	2.9726	3.7528	2.9	0.4863	1.3727

Table 2: Results of calculations of *Q* and H_0 values when varying H_1 and H_2

H_1	Q	H_0	H_2	Q	H_0
0.9	1.1120	1.8292	1.09	1.3965	1.5533
1.0	1.1554	1.7755	1.19	1.3627	1.5775
1.1	1.1928	1.7161	1.29	1.2425	1.7051
1.2	1.2235	1.6494	1.39	1.1598	1.7695
1.3	1.2460	1.5728	1.40	1.1634	1.7694

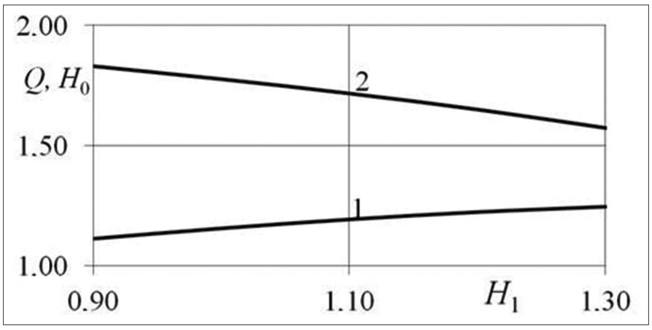

Figure 3: Dependence of the cofferdam flow rate Q and the ordinate H_0 of the free surface outlet point H_0 on the filter length H_1

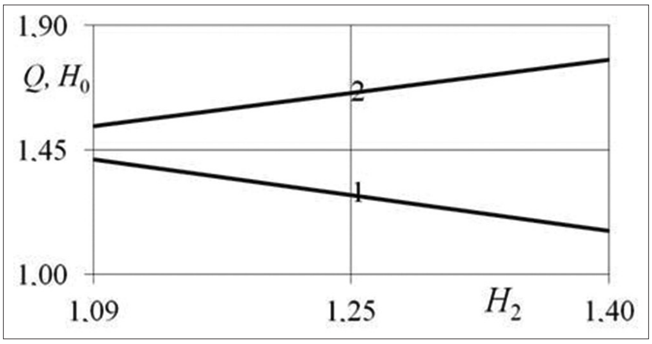

Figure 4: Dependence of the cofferdam flow rate Q and the ordinate H_0 of the free surface outlet point on the water level in the down streamer servoir H_2

- As the width of the coffer dam *L* increases, the flow rate *Q* and the ordinate H_0 of the free surface exit point to the screen decrease.

From Table 2 and Figures 3 and 4 follows that decrease of parameters H_1 and H_2 by 1.5 and 1.3 times, respectively, leads to change of *Q* value by 16.8% (at fixing H_1) and 12% (at fixing H_2). The marked regularities lead to the conclusion that the cofferdam flow rate depends on the value of level lowering to a somewhat greater extent than on the filter length (or on imperfection of a well or a well).

For the base case, almost all the dependences of Q and H_0 on the parameters ε, H, H_1, H_2, and L are close to linear.

Comparison of the exact values obtained for the base case Q = 1.155 and H_0 = 1.776 with the approximate values Q = 1.141 and H_0 = 1.768 for the base case[9] where the flow area to the left was limited by the equipotential shows that the relative error of the calculations is rather small and amounts to only 0.5 and 1.3% respectively.

Comparison of exact value of flow Q = 1.16, obtained for basic variant, with approximate value Q =1.26, which follows at application of generalized formula of I.A. Charny [1, p. 267] for usual rectangular cofferdam (without screen) in the presence of evaporation

$$Q = -\frac{\varepsilon L}{2} + \frac{H^2 - H_2^2}{2L},$$

Leads to an error of 8.3%.

For comparison with data $H=1$, $H_1=0.05$, $H_2=0.238$, $L=4$ work[7] at absence of evaporation, that is, at ε = 0, for which values Q = 0.118, $H_0=0.29$ are received by the approximate formulas in semi-inverse formulation, we consider variant ε = 0.1, $H=1$, $H_1=0.05$, $H_2=0.238$, $L=4$, leading to exact values Q = 0.42, H_0 = 0.75. Here, relative calculation errors are 71 and 61%, respectively. Consequently, Just as in,[9] evaporation significantly affects the flow pattern.

CONCLUSION

The method for construction of exact analytical solution of a problem about movement of liquid in a rectangular cofferdam with a screen in the presence of evaporation from a free surface of ground waters has been developed. The investigation shows that the filtration scheme in a rectangular cofferdam with impermeable screen, firstly, is very similar to the previously considered[9] problem about movement of ground waters to the imperfect gallery, one of them being limiting with respect to the other. Second, the flow pattern near the screen essentially depends not only on the filter size, but also on the presence of evaporation, which strongly affects the flow rate value and the ordinate of the outlet point of the depression curve on the screen. The obtained results, announced in,[16] give so meidea (at least qualitatively) about possible dependence of motion characteristics when considering the filtration problem already to imperfect well or tubular well.

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